Dynamics of a population of interacting plant and pollinator species

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Sylvie's birthday - 2018

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Motivations

★ Bees play a major role in the pollination of plants.

★ Fear that the extinction of some bees species ends in a cascade of exinction in bees and plants species.

★ We consider a bipartite network of bee and plant species:

- fixed in time
- model population dynamics with interactions along the graph

Will a large complex system be stable? (R. May, 1972)

Ecological stability by May (1972)

★ Consider a graph with n vertices (species) and connectance

$$C = \frac{|E|}{n(n-1)}.$$

★ Assume that around the equilibrium:

$$N_t^i = \widehat{N}^i + x_t^i, \qquad \text{with} \qquad \frac{dx^i}{dt} = \sum_{j=1}^{n} a_{ij} x_j(t),$$

where

- $a_{ii} = -1$,
- $a_{ij} = 0$ if there is no edge between species i and j,
- a_{ij} is an independent centered r.v. with variance s^2, if $i \sim j$.

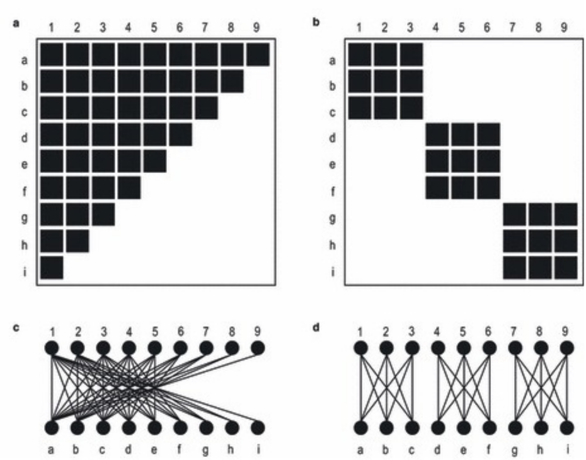
★ The system is stable if the eigenvalues of (a_{ij}) are negative.

May says that an equilibrium of the population is stable if and only if

$$s\sqrt{nC} < 1.$$

Shapes of plant-pollinator graphs

★ The networks are sometimes distinguished by nestedness and connectance.



Fontaine, Guimaraes, Kéfi, ... Thébault (2011)

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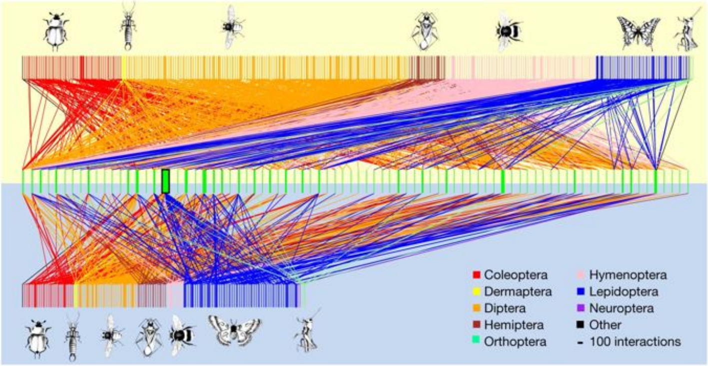

Knop, Zoller, Ryser, Gerpe, Hörler, Fontaine (2017)

Model of pollinator and plant

★ Structuration by **degree of specialization** of the species, x (plants) or y (bees) in $[0,1]$.

★ We have n populations of plants and m populations of pollinators.

$$\mathbf{P}_t^n(dx) = \frac{1}{n}\sum_{i=1}^{n} P_t^i \delta_{x^i}(dx), \qquad \mathbf{A}_t^m(dy) = \frac{1}{m}\sum_{j=1}^{m} A_t^j \delta_{y^j}(dy).$$

★ Rescaling parameter K:

$$\mathbf{P}_t^{K,n}(dx) = \sum_{i=1}^{n} \frac{1}{nK} P_t^{K,i} \delta_{x^i}(dx), \qquad \mathbf{A}_t^{K,m}(dy) = \sum_{j=1}^{m} \frac{1}{mK} A_t^{K,j} \delta_{y^j}(dy).$$

→ Three parameters: K, n, m

Modelling a bipartite graph

★ **Vertices:** each pollinator and plant species.

★ **Edges:** $i \sim j$ if the pollinator species j visits the plant species i.

d_A^j and d_P^i are the degrees of the species j and i.

★ **Graph:** The bipartite graph $\mathcal{G}$ can be represented by an $n \times m$ adjacency matrix $G^{n,m}$, with $G_{ij}^{n,m} = 1$ if the pollinator specie j interacts with the plant specie i, i.e. $i \sim j$, and 0 otherwise.

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★ **Bipartite Erdös-Rényi graphs:** Each plant species i and pollinator species j are connected with the probability $\phi(x^i, y^j)$ independently of the other couples.

Examples: $\phi(x, y) = xy$ or $\phi(x, y) = \varphi(|x - y|)$.

Interactions along the graph

★ Growth rate=natural birth - natural death rates:

$$g^P(r) = \frac{\alpha_P\ r}{\beta_P + \gamma_P\ r} - (d_3 + d_4 r), \qquad g^A(r) = \frac{\alpha_A\ r}{\beta_A + \gamma_A\ r} - (d_1 + \frac{d_2}{r})$$

where r is the resources that a given species can obtain from its interactions.

★ $c_{ij}^{n,m}$ is a weight for the interaction $i \sim j$:

$$R_t^{i,nm} = \sum_{j \sim i} c_{ij}^{n,m} A_t^j P_t^i.$$

★ A death by competition for the bees:

$$\sum_{\ell=1}^{m} H(y, y^\ell) A_t^\ell.$$

Semi-martingales

★ for a test function f:

$$\langle \mathbf{P}_t^{K,n}, f\rangle = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{K}P_t^{K,i}f(x_i)$$

$$= \langle \mathbf{P}_0^{K,n}, f\rangle + \int_0^t \frac{1}{n}\sum_{i=1}^{n} f(x_i)g^P\left(R_s^{K,i,nm}\right)\frac{1}{K}P_s^{K,i}\, ds + \sum_{i=1}^{n} f(x_i)M_t^{K,n,i,f},$$

★ where $M^{K,n,i,f}$'s are square integrable martingales with

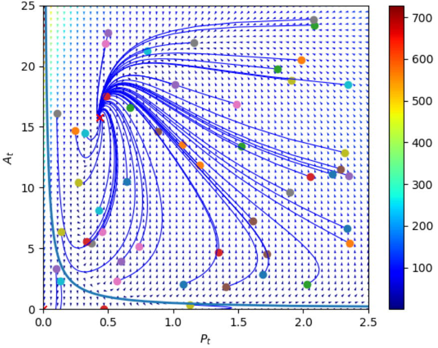
$$\langle M^{K,i,f}\rangle_t = \frac{1}{n^2K}\sum_{i=1}^{n} f^2(x_i)\int_0^t \left(b^P(R^{K,i,nm}) + d^P(R^{K,i,nm})\right)\frac{1}{K}P_s^{K,i}\, ds.$$

Large lopulation limit

★ Large population limit for n, m fixed.

$$\frac{dP^i_t}{dt} = g^P\left(P^i_t \sum_{j\sim i} c^{n,m}_{ij} A^j_t\right) P^i_t$$

$$\frac{dA^j_t}{dt} = \left[g^A\left(A^j_t \sum_{i\sim j} c^{n,m}_{ij} P^i_t\right) - \sum_{\ell=1}^{m} H_m(y^j, y^\ell) A^\ell_t\right] A^j_t,$$



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$$\frac{dP_t^i}{dt} = g^P\left(P_t^i \sum_{j\sim i} c_{ij}^{n,m} A_t^j\right) P_t^i$$

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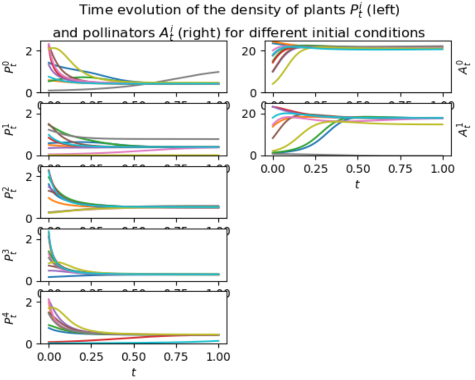
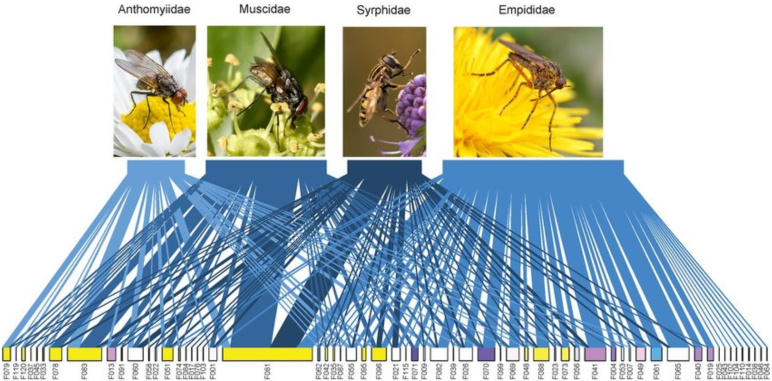


Large networks

- ★ Erdös-Rényi graphs belong to the family of **dense** random graphs.

- ★ When $n, m \to +\infty$, the graph converge to the **graphon** $\phi \ : \ [0,1]^2 \mapsto [0,1]$



Lefebvre, Villemant, Fontaine, Daugeron (2018)

Graphon convergence

★ For a 'small' motif F, let us define:

$$t(F,G) = \frac{|\mathrm{inj}(F,G)|}{(n)_k}, \qquad (n)_k = n(n-1)\cdots(n-k+1) = \frac{n!}{k!}$$

★ The class $\mathcal{F}$ of isomorphism classes of finite graphs is denumberable and we denote by $(F_i)_{i\in\mathbb{N}}$ an enumeration.

★ We define a distance between graphs by:

$$d(G,G') = \sum_{i\in\mathbb{N}} 2^{-i}|t(F_i,G) - t(F_i,G')|.$$

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★ We can extend to the distance between G and a graphon ϕ with:

$$t(F,\phi) = \int_{[0,1]^k} \prod_{\{i,j\}\in E(F)} \phi(x_i,x_j)dx_1\dots dx_k.$$

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Def: A sequence of graphs $(\mathcal{G}_n)_{n\in\mathbb{N}}$ convergences to the graphon ϕ if

$$\lim_{n\to+\infty} d(G_n,\phi) = 0.$$

Modelling the ressources

Recall:

$$R_t^{i,nm} = \sum_{j \sim i} c_{ij}^{n,m} A_t^j P_t^i.$$

★ **Hyp:** it is possible to index the plant and pollinator species s.t. for all $n, m \in \mathbb{N}^*$, there exists a continuous (random) function $c^{n,m} : [0,1]^2 \mapsto \mathbb{R}$, s.t.

$$\forall i \in \{1, \dots, n\},\ \forall j \in \{1, \dots, m\},\ c_{ij}^{n,m} = c^{n,m}\left(\frac{i}{n}, \frac{j}{m}\right).$$

Moreover, we suppose that there exists a (random) function $c : [0,1]^2 \mapsto \mathbb{R}$ such that the sequence of functions $c^{n,m}(.,.)$ converges to $c(.,.)$, when n and m tend to $+\infty$.

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★ **Example:** when the $c_{ij}^{n,m}$ depend on the degree of generalism:

$$c_{ij}^{n,m} = \rho\left(x^i, y^j\right) = \rho\Big(F_{P,n}^{-1}\left(\frac{i}{n}\right), F_{A,m}^{-1}\left(\frac{j}{n}\right)\Big).$$

Cinetic averaging of the graph

★ **Th:** Let us consider two independent sequences of i.i.d. random variables $(x^i)_{i\geq 1}$, $(y^j)_{j\geq 1}$ on $[0,1]$, and good ICs. For $n, m \to +\infty$, $(\mathbf{P}^n_t(dx), \mathbf{A}^m_t(dx))_{t\geq 0}$ converge in $\mathcal{C}(\mathbb{R}_+, \mathcal{M}^2_F([0,1]))$ to a process $(\bar{P}, \bar{A})$ in $\mathcal{C}(\mathbb{R}_+, \mathcal{M}^2_F([0,1]))$, such that:

(i) for all $t \geq 0$, $\bar{P}_t$ and $\bar{A}_t$ admit densities $\bar{p}(x,t)$ and $\bar{a}(y,t)$ with respect to the Lebesgue measure on $[0,1]$,

(ii) $(\bar{P}, \bar{A})$ is the unique solution, for $f \in \mathcal{C}([0,1], \mathbb{R})$, of

$$\begin{aligned}\int_0^1 f(x)d\bar{P}_t(x) = &\int_0^1 f(x)d\bar{p}_0(x)dx \\ &+ \int_0^t \int_0^1 f(x)g^P\Big(\bar{p}(x,s)\int_0^1 \rho(x,y)\phi(x,y)a(y,s)dy\Big)\bar{p}(x,s)dx\ ds,\end{aligned}$$

$$\begin{aligned}\int_0^1 f(y)d\bar{A}_t(y) = &\int_0^1 f(y)d\bar{a}_0(y)dy \\ &+ \int_0^t \int_0^1 f(y)\Big[g^A\Big(\bar{a}(x,s)\int_0^1 \rho(x,y)\phi(x,y)\bar{p}(x,s)dx\Big) \\ &-H * \bar{a}(y,s)\Big]\bar{a}(y,s)dy\ ds.\end{aligned}$$

Thank You