Limit theorems for Hawkes processes with finite-range self-excitation and self-inhibition

Populations: Interactions and Evolution
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[5] Former PhD student of Sylvie
[6] Met Sylvie in Marc Yor's masters course, we coauthored 9 papers in international journals

Introduction

Hawkes processes were introduced to model earthquakes and their replicas.[1]
These point processes are now used in a increasingly wide variety of applicative fields to model

the arrival instants of events

exhibiting a form of

self-excitation, or attraction:
the occurrence of an event tends to
encourage the occurrence of subsequent events.

---

1 A. Hawkes (1971). “Spectra of some self-exciting and mutually exciting point processes”. In: *Biometrika* 1.

Many such fields need to model phenomena exhibiting also

self-inhibition, or repulsion:
the occurrence of an event tends to
discourage the occurrence of subsequent events.

For instance in neurobiology the firing of a neuron may create a refractory period for this neuron, and complex regulations are obtained by some neurons having an inhibitory effect on others.

The case of

self-excitation

has been studied actively for a long time, and is well documented. Notably, Hawkes and Oakes[2] have provided a decomposition as a

cluster point process with immigration and branching: the points which arise from the excitation from a previous point are considered as the offspring of that ancestor.

It allows to apply branching process techniques.

[2] A.G. Hawkes and D. Oakes (1974). "A cluster process representation of a self-exciting process". In: *Journal of Applied Probability*.

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cluster point process with immigration and branching: the points which arise from the excitation from a previous point are considered as the offspring of that ancestor.

It allows to apply branching process techniques.
In contrast,

self-inhibition

is much less well understood, in particular due to the loss of monotonicity.

Notably, the cluster point process representation fails.

[2] A.G. Hawkes and D. Oakes (1974). "A cluster process representation of a self-exciting process". In: *Journal of Applied Probability*.

The purpose of our paper[3] is to consider processes with both

self-inhibition and self-excitation

and obtain

long-time limit results
suitable for statistical applications,

and in particular to extend

concentration inequalities

obtained by Reynaud-Bouret and Roy[4].

---

[3] M. Costa et al. (2018). "Renewal in Hawkes processes with self-excitation and inhibition". In: *arXiv:1801.04645*.

[4] P. Reynaud-Bouret and E. Roy (2007). "Some non asymptotic tail estimates for Hawkes processes". In: *Bulletin of the Belgian Mathematical Society-Simon Stevin* 5.

Definition of the Hawkes process

Hawkes process

Definition 1

The point process N^h on $\mathbb{R}$ is a Hawkes process on $(0, +\infty)$ with

- initial condition $N^0 \in \mathcal{N}((-\infty, 0])$ with law $\mathfrak{m}$,
- base intensity $\lambda > 0$,
- and (signed) reproduction function $h : (0, +\infty) \to \mathbb{R}$,

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- and (signed) reproduction function $h : (0,+\infty) \to \mathbb{R}$,

if $N^h|_{(-\infty,0]} = N^0$ and the conditional intensity of $N^h|_{(0,+\infty)}$ w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ is given by

$$\Lambda^h : t \in (0,+\infty) \mapsto \Lambda^h(t) = \left(\lambda + \sum_{u\in N^h,\, u<t} h(t-u)\right)^+$$

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if $N^h|_{(-\infty,0]} = N^0$ and the conditional intensity of $N^h|_{(0,+\infty)}$ w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ is given by

$$\begin{aligned} \Lambda^h : t \in (0,+\infty) \mapsto \Lambda^h(t) &= \Big(\lambda + \sum_{u\in N^h,\, u<t} h(t-u)\Big)^+ \\ &= \Big(\lambda + \int_{(-\infty,t)} h(t-u)\, N^h(du)\Big)^+ . \end{aligned} \tag{1}$$

Martingale formulation

Definition 1 is a

martingale formulation of an equation for the law of N^h: the conditional intensity Λ^h of N^h depends on N^h itself.

Existence and uniqueness (in law) of the Hawkes process has to be proved, under appropriate assumptions.

Poisson-driven equation representation

We take a unit $(\mathscr{F}_t)_{t\geq 0}$-Poisson point process Q on $(0,+\infty)^2$.
We build a pathwise unique strong solution of the equation

$$\begin{cases} N^h = N^0 + \displaystyle\int_{(0,+\infty)\times(0,+\infty)} \delta_u \mathbb{1}_{\{\theta\leq\Lambda^h(u)\}}\, Q(\mathrm{d}u,\mathrm{d}\theta)\,, \\ \Lambda^h(u) = \left(\lambda + \displaystyle\int_{(-\infty,u)} h(u-s)\, N^h(\mathrm{d}s)\right)^+, \quad u>0\,. \end{cases} \tag{2}$$

Such equations have been much studied when $h \geq 0$, see[5], *e.g.*

[5] P. Brémaud and L. Massoulié (1996). "Stability of nonlinear Hawkes processes". In: *Annals of Probability* 3; P. Brémaud, G. Nappo, and G.L. Torrisi (2002). "Rate of convergence to equilibrium of marked Hawkes processes". In: *Journal of Applied Probability*; L. Massoulié (1998). "Stability results for a general class of interacting point processes dynamics, and applications". In: *Stochastic Processes and their Applications*.

Existence and uniqueness

Existence and uniqueness is a non-explosion issue.

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The counting process $(N_t^h)_{t\in\mathbb{R}}$ of N^h is given by

$$N^h((a,b]) = N_b^h - N_a^h\,, \quad -\infty < a < b < \infty\,, \qquad N_0^h = 0\,.$$

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$$N^h((a,b]) = N^h_b - N^h_a\,,\quad -\infty < a < b < \infty\,, \qquad N^h_0 = 0\,.$$

It satisfies the time-inhomogeneous SDE, equivalent to (2),

$$\begin{cases} N^h_t = \displaystyle\int_{(0,t]\times(0,+\infty)} \mathbb{1}_{\{\theta\le\Lambda^h(u)\}}\, Q(\mathrm{d}u,\mathrm{d}\theta)\,, & t\ge 0\,, \\ \Lambda^h(u) = \Bigg(\lambda + \underbrace{\displaystyle\int_{(-\infty,0]} h(u-s)\,\mathrm{d}N^h_s}_{\text{determined by } N^0} + \underbrace{\displaystyle\int_{(0,u)} h(u-s)\,\mathrm{d}N^h_s}_{\text{depends on } (N^h_s)_{0\le s\le u}}\Bigg)^+, & u>0\,. \end{cases}$$

Notation

When the Hawkes process is well-defined, *i.e.*, when existence and uniqueness holds, we use the notation

$$\mathbb{P}_{\mathfrak{m}} \quad \text{and} \quad \mathbb{E}_{\mathfrak{m}}$$

to specify that N^0 has law $\mathfrak{m}$.

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When

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we use

$$\mathbb{P}_\nu \quad \text{and} \quad \mathbb{E}_\nu \,.$$

We shall often consider the case

$$\nu = \emptyset \triangleq \text{the null measure having no point on } (-\infty, 0] \,.$$

Range of influence

The support of h is naturally defined as the support of the measure

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We assume w.l.o.g. that $h = h\mathbb{1}_{\mathrm{supp}(h)}$ and define

$$L(h) \triangleq \sup(\mathrm{supp}(h)) \triangleq \sup\{t > 0, |h(t)| > 0\} \in [0,+\infty]\,.$$

This is the **maximal range of influence of a point**.

Main results

Shifted point processes

We aim at studying the limit behavior of the process on

a sliding finite time window of length A.

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For a point process N we introduce the **time-shifted processes**

$$N(\cdot + t) \equiv (N_{t+s} - N_t)_{s\in\mathbb{R}}\,, \qquad t \geq 0\,,$$

which is such that

$$N^h(\cdot + t)((a,b]) = N^h((a+t, b+t])\,, \quad -\infty < a < b < \infty\,.$$

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which is such that

$$N^h(\cdot + t)((a,b]) = N^h((a+t, b+t])\,, \quad -\infty < a < b < \infty\,.$$

Time is labeled so that observation has started by time $-A$, and

$$N^h(\cdot + t)|_{(-A,0]} = N^h|_{(t-A,t]}(\cdot + t) \equiv (N_{t+s} - N_t)_{-A<s\leq 0}$$

where we abuse notation by identifying $N|_B$ and $N\mathbb{1}_B$, etc.

Quantities of interest

The quantities of interest will be of the form

$$\frac{1}{T}\int_0^T f\left(N^h(\cdot + t)|_{(-A,0]}\right) \mathrm{d}t \tag{3}$$

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- $A > 0$ a finite sliding window length,
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- $T > 0$ is a large time horizon, and we shall let $T \to \infty$.

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- f is a locally bounded Borel function on $\mathcal{N}((-A,0])$,
- $T > 0$ is a large time horizon, and we shall let $T \to \infty$.

Such quantities appear commonly in the field of

statistical inference of random processes.

Recall that observation has started by time $-A$.

Main assumptions

Assumption 1

The signed function $h : (0, +\infty) \longrightarrow \mathbb{R}$ is such that

$$L(h) \le A < \infty\,, \qquad \|h^+\|_1 \triangleq \int_{(0,+\infty)} h^+(t)\,\mathrm{d}t < 1\,.$$

The distribution $\mathfrak{m}$ of the initial condition N^0 is such that

$$\mathbb{E}_{\mathfrak{m}}\big(N^0(-L(h), 0]\big) < \infty.$$

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Then the quantities (3) actually depend only on the restriction $N^0|_{(-A,0]}$ of the initial condition N^0 to $(-A, 0]$.
With abuse of notation, we identify $\mathfrak{m}$ with its marginal on $\mathcal{N}((-A, 0])$ and denote by $\emptyset$ the null point process on $(-A, 0]$.

Auxiliary Markov process

To exploit the regeneration structure of the Hawkes process N^h with finite influence range, we introduce the auxiliary process

$$(X_t)_{t\geq 0}\,, \quad X_t \triangleq N^h(\cdot + t)|_{(-A,0]}\,.$$

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- the process $(X_t)_{t\geq 0}$ is strong Markov for $(\mathscr{F}_t)_{t\geq 0}$,
- the null point process $\emptyset$ is positive recurrent,
- there exists a unique invariant law π_A.

Note that we can then construct a two-sided Markov process in equilibrium on $\mathbb{R}$, and hence a stationary version of N^h on $\mathbb{R}$.

Strong Markov property

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- the fact that Q is a $(\mathscr{F}_t)_{t\geq 0}$-Poisson point process and thus satisfies the strong Markov property,
- existence and uniqueness in law of the solution of Eq. (2).

Invariant law

The invariant law π_A of $(X_t)_{t\geq 0}$ can be written in terms of the occupation measure over an excursion.

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The return time to $\emptyset$ is defined by

$$\begin{aligned}\tau &\triangleq \inf\{t > 0 : X_{t-} \neq \emptyset, X_t = \emptyset\} \\ &\triangleq \inf\{t > 0 : N^h[t-A, t) \neq 0, N^h(t-A, t] = 0\}\,.\end{aligned}$$

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For every non-negative Borel function f,

$$\begin{aligned}\pi_A f &\triangleq \frac{1}{\mathbb{E}_\emptyset(\tau)}\mathbb{E}_\emptyset\left(\int_0^\tau f(X_t)\,\mathrm{d}t\right)\\ &\triangleq \frac{1}{\mathbb{E}_\emptyset(\tau)}\mathbb{E}_\emptyset\left(\int_0^\tau f\left(N^h(\cdot + t)|_{(-A,0]}\right)\mathrm{d}t\right).\end{aligned} \tag{4}$$

Pointwise ergodic theorem

Theorem 1

Let N^h be a Hawkes process with $\lambda > 0$, $h : (0, +\infty) \longrightarrow \mathbb{R}$, and N^0 with law $\mathfrak{m}$, satisfying Assumption 1.

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Let N^h be a Hawkes process with $\lambda > 0$, $h : (0, +\infty) \longrightarrow \mathbb{R}$, and N^0 with law $\mathfrak{m}$, satisfying Assumption 1.
Then it has an unique invariant law π_A given by (4).
If f is a Borel function which is nonnegative or π_A-integrable, then

$$\frac{1}{T}\int_0^T f\left(N^h(\cdot + t)|_{(-A,0]}\right) \mathrm{d}t \xrightarrow[T\to\infty]{\mathbb{P}_{\mathfrak{m}}-\text{a.s.}} \pi_A f \,.$$

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We use renewal techniques to prove this.

Convergence to a process in equilibrium

Theorem 2

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Then convergence to equilibrium for large times holds in the following sense:

$$\mathbb{P}_{\mathfrak{m}}\big(N^h(\cdot + t)|_{[0,+\infty)} \in \cdot\big) \xrightarrow[t\to\infty]{\text{total variation}} \mathbb{P}_{\pi_A}(N^h|_{[0,+\infty)} \in \cdot)\,.$$

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This is used in proofs of the sequel, and has independent interest.

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This is used in proofs of the sequel, and has independent interest.
We use results in Thorisson[6] based on renewal techniques and coupling to prove it.

6 Hermann Thorisson (2000). *Coupling, stationarity, and regeneration.*

Central limit theorem

Theorem 3

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If f is Borel and π_A-integrable and satisfies

$$\sigma^2(f) \triangleq \frac{1}{\mathbb{E}_\emptyset(\tau)} \mathbb{E}_\emptyset\left(\left(\int_0^\tau \left(f\left(N^h(\cdot + t)|_{(-A,0]}\right) - \pi_A f\right) \mathrm{d}t\right)^2\right) < \infty$$

then

$$\sqrt{T}\left(\frac{1}{T}\int_0^T f\left(N^h(\cdot + t)|_{(-A,0]}\right) \mathrm{d}t - \pi_A f\right) \xrightarrow[T\to\infty]{\text{in law}} \mathcal{N}(0, \sigma^2(f))\,.$$

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$$\sqrt{T}\left(\frac{1}{T}\int_0^T f\left(N^h(\cdot + t)|_{(-A,0]}\right) \mathrm{d}t - \pi_A f\right) \xrightarrow[T\to\infty]{\text{in law}} \mathcal{N}(0, \sigma^2(f))\,.$$

We use renewal techniques to prove this.

Pointwise ergodic theorems and their CLTs have been much investigated in the case of nonnegative reproduction functions h, see[7], *e.g.* The results usually concern the

instantaneous values N_t^h of the counting process.

The proofs usually rely on martingale techniques.

[7] E. Bacry et al. (2013). "Some limit theorems for Hawkes processes and application to financial statistics". In: *Stochastic Process. Appl.* 7.

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instantaneous values N_t^h of the counting process.

The proofs usually rely on martingale techniques.

Here the results concern sliding windows of arbitrary finite length of the point measure N^h, and hence the

sub-processes $(N_{t+s}^h)_{-A<s\leq 0}$ of the counting process.

The proofs use renewal techniques that will also help us establish

non-asymptotic exponential concentration bounds.

[7] E. Bacry et al. (2013). "Some limit theorems for Hawkes processes and application to financial statistics". In: *Stochastic Process. Appl.* 7.

Notation

Let the first entrance time at $\emptyset$ be defined by

$$\tau_0 \triangleq \inf\{t \geq 0 : N^h(t-A,t] = 0\}\,. \tag{5}$$

Recall that $x^+ = \max(x,0)$ and $x^- = \max(-x,0)$ for $x \in \mathbb{R}$.
Let $(x)_\pm^k = (x^\pm)^k$ and

$$c^\pm(f) \triangleq \sup_{k\geq 3}\left(\frac{2}{k!}\frac{\mathbb{E}_\emptyset\left(\left(\int_0^\tau \left(f\left(N^h(\cdot+t)|_{(-A,0]}\right)-\pi_A f\right)\,\mathrm{d}t\right)_\pm^k\right)}{\mathbb{E}_\emptyset(\tau)\sigma^2(f)}\right)^{\frac{1}{k-2}},$$

$$c^\pm(\tau) \triangleq \sup_{k\geq 3}\left(\frac{2}{k!}\frac{\mathbb{E}_\emptyset\left((\tau-\mathbb{E}_\emptyset(\tau))_\pm^k\right)}{\mathrm{Var}_\emptyset(\tau)}\right)^{\frac{1}{k-2}},$$

$$c^+(\tau_0) \triangleq \sup_{k\geq 3}\left(\frac{2}{k!}\frac{\mathbb{E}_{\mathfrak{m}}\left((\tau_0-\mathbb{E}_{\mathfrak{m}}(\tau_0))_+^k\right)}{\mathrm{Var}_{\mathfrak{m}}(\tau_0)}\right)^{\frac{1}{k-2}}.$$

Concentration inequalities

Theorem 4

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Let N^h be a Hawkes process with $\lambda > 0$, $h : (0, +\infty) \longrightarrow \mathbb{R}$, and N^0 with law $\mathfrak{m}$, satisfying Assumption 1.
If f is Borel and takes its values in a bounded interval $[a, b]$ then, for all $\varepsilon > 0$ and T sufficiently large,

$$\mathbb{P}_{\mathfrak{m}}\left(\left|\frac{1}{T}\int_0^T f\left(N^h(\cdot + t)|_{(-A,0]}\right) \mathrm{d}t - \pi_A f\right| \geq \varepsilon\right)$$

satisfies a concentration inequality bound depending only on the parameters, given in the next slide.

Concentration inequalities

Theorem 4

Let N^h be a Hawkes process with $\lambda > 0$, $h : (0, +\infty) \longrightarrow \mathbb{R}$, and N^0 with law $\mathfrak{m}$, satisfying Assumption 1.
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satisfies a concentration inequality bound depending only on the parameters, given in the next slide.

We used renewal techniques and the reference book[8] in the proof.

8 Pascal Massart (2007). *Concentration inequalities and model selection*.

$$
\begin{aligned}
&\mathbb{P}_{\mathfrak{m}}\left(\left|\frac{1}{T}\int_0^T f\big(N^h(\cdot+t)|_{(-A,0]}\big)\,\mathrm{d}t-\pi_A f\right|\geq\varepsilon\right)\\
&\quad\leq \exp\left(-\frac{((T-\sqrt{T})\varepsilon-|b-a|\mathbb{E}_\emptyset(\tau))^2}{8T\sigma^2(f)+4c^+(f)((T-\sqrt{T})\varepsilon-|b-a|\mathbb{E}_\emptyset(\tau))}\right)\\
&\qquad+\exp\left(-\frac{((T-\sqrt{T})\varepsilon-|b-a|\mathbb{E}_\emptyset(\tau))^2}{8T\sigma^2(f)+4c^-(f)((T-\sqrt{T})\varepsilon-|b-a|\mathbb{E}_\emptyset(\tau))}\right)\\
&\qquad+\exp\left(-\frac{((T-\sqrt{T})\varepsilon-|b-a|\mathbb{E}_\emptyset(\tau))^2}{8T|b-a|^2\frac{\mathrm{Var}_\emptyset(\tau)}{\mathbb{E}_\emptyset(\tau)}+4|b-a|c^+(\tau)((T-\sqrt{T})\varepsilon-|b-a|\mathbb{E}_\emptyset(\tau))}\right)\\
&\qquad+\exp\left(-\frac{((T-\sqrt{T})\varepsilon-|b-a|\mathbb{E}_\emptyset(\tau))^2}{8T|b-a|^2\frac{\mathrm{Var}_\emptyset(\tau)}{\mathbb{E}_\emptyset(\tau)}+4|b-a|c^-(\tau)((T-\sqrt{T})\varepsilon-|b-a|\mathbb{E}_\emptyset(\tau))}\right)\\
&\qquad+\exp\left(-\frac{(\sqrt{T}\varepsilon-2|b-a|\mathbb{E}_{\mathfrak{m}}(\tau_0))^2}{8|b-a|^2\,\mathrm{Var}_{\mathfrak{m}}(\tau_0)+4|b-a|c^+(\tau_0)(\sqrt{T}\varepsilon-2|b-a|\mathbb{E}_{\mathfrak{m}}(\tau_0))}\right).
\end{aligned}
$$

If $N|_{(-A,0]}=\emptyset$ then the last term of the r.h.s. is null and the upper bound is true with T instead of $T-\sqrt{T}$ in the other terms.

More practical exponential bounds

This concentration inequality can be simplified using upper bounds for the constants $c^{\pm}(f)$ and $c^{\pm}(\tau)$.

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Theorem 5

Under these assumptions, there exists $\alpha > 0$ s.t. $\mathbb{E}_{\emptyset}(e^{\alpha\tau}) < \infty$. Let

$$v = \frac{2(b-a)^2}{\alpha^2} \left\lfloor \frac{T}{\mathbb{E}_{\emptyset}(\tau)} \right\rfloor \mathbb{E}_{\emptyset}(e^{\alpha\tau}) e^{\alpha \mathbb{E}_{\emptyset}(\tau)} \quad \text{and} \quad c = \frac{|b-a|}{\alpha}.$$

Then for all $\varepsilon > 0$ we can give a more practical expression for

$$\mathbb{P}_{\mathfrak{m}}\left(\left| \frac{1}{T} \int_0^T f\left(N^h(\cdot + t)|_{(-A,0]}\right) \mathrm{d}t - \pi_A f \right| \geq \varepsilon \right).$$

See the next slide.

Exponential bounds

In particular

$$\mathbb{P}_\emptyset\left(\left|\frac{1}{T}\int_0^T f\left(N^h(\cdot+t)|_{(-A,0]}\right)\mathrm{d}t - \pi_A f\right| \geq \varepsilon\right)$$
$$\leq 4\exp\left(-\frac{\left(T\varepsilon - |b-a|\mathbb{E}_\emptyset(\tau)\right)^2}{4\left(2v + c(T\varepsilon - |b-a|\mathbb{E}_\emptyset(\tau))\right)}\right)$$

or equivalently, for all $1 \geq \eta > 0$,

$$\mathbb{P}_\emptyset\left(\left|\frac{1}{T}\int_0^T f\left(N^h(\cdot+t)|_{(-A,0]}\right)\mathrm{d}t - \pi_A f\right| \geq \varepsilon_\eta\right) \leq \eta\,,$$

$$\varepsilon_\eta = \frac{1}{T}\left(|b-a|\mathbb{E}_\emptyset(\tau) - 2c\log\left(\frac{\eta}{4}\right) + \sqrt{4c^2\log^2\left(\frac{\eta}{4}\right) - 8v\log\left(\frac{\eta}{4}\right)}\right)\,.$$

Main ideas for the proofs

Coupling with a Hawkes process with pure self-excitation

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and in particular to prove positive recurrence.

Existence and uniqueness, coupling

Theorem 6

Consider Equation (2) for N^h and the similar equation for N^{h^+} in which h is replaced by h^+. Assume that

$$\|h^+\|_1 < 1\,, \quad \forall t > 0,\ \int_0^t \mathbb{E}_{\mathfrak{m}}\Big(\int_{(-\infty,0]} h^+(u-s)\,N^0(\mathrm{d}s)\Big)\,\mathrm{d}u < \infty.$$

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Then there exists a pathwise unique strong solution N^h which is a Hawkes process in the sense of Definition 1, the similar result holds for N^{h^+}, and moreover

$$N^h \le N^{h^+}\,.$$

Exponential moments for the return time

Recall that the return time to $\emptyset$ is given by

$$\tau \triangleq \inf\{t > 0 : N^h[t - A, t) \neq 0, N^h(t - A, t] = 0\},$$

We shall prove the existence of exponential moments

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Indeed, denoting by τ^+ the corresponding return time for the process N^{h^+} defined with $h^+ \geq 0$, clearly

$$\tau \leq \tau^+$$

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and we need only study τ^+ and N^{h^+}, *i.e.*, Hawkes processes with non-negative reproduction functions.

Cluster point process decomposition

A Hawkes process with **non-negative** reproduction function h enjoys a **decomposition as a cluster point process with immigration and branching**.[9]

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- The conditional intensity can be decomposed into the sum

$$\Lambda^h(t) = \lambda + \sum_{u \in N^h,\, u<t} h(t-u)\,.$$

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- The arrivals due to the influence of a previous arrival are considered as those of its offspring.

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- The arrivals due to the base intensity λ are considered as those of immigrants.
- The arrivals due to the influence of a previous arrival are considered as those of its offspring.
  Such arrivals are i.i.d. up to time-shift, and belong to a multi-generational family started by an immigrant.

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In this context $\|h\|_1$ is interpreted as the

mean number of offspring of any arrival

and

$$\|h\|_1 \leq 1$$

is the classic sub-criticality condition which ensures that each family started by an immigrant is finite and well-controlled.

Interpretation using a $M/G/\infty$ queue

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Recall that the influence range of a point satisfies

$$L(h) \leq A < \infty \,.$$

Thus, after this service time we are sure that the job and its family will not influence the future of N^h.

Emptying of the $M/G/\infty$ queue

Let $(Y_t)_{t\geq 0}$ denote the queue-length process, $\mathcal{T}_0 \triangleq 0$, and

$$\mathcal{T}_k \triangleq \inf\{t > \mathcal{T}_{k-1} : Y_{t-} \neq 0\,,\ Y_t = 0\}\,, \quad k \geq 1\,.$$

Then the $(\mathcal{T}_k)_{k\geq 1}$ are renewal times for the auxiliary Markov process $(X_t)_{t\geq 0}$,

$$X_{\mathcal{T}_k} = \emptyset \;\text{ for }\; k \geq 1\,, \quad \text{and} \;\; \tau \leq \mathcal{T}_1\,.$$

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We prove positive recurrence of $\emptyset$ for $(X_t)_{t\geq 0}$ by proving positive recurrence of 0 for $(Y_t)_{t\geq 0}$. We moreover prove that

$$\mathbb{E}_\emptyset(e^{\alpha \mathcal{T}_1}) < \infty \ \text{ for some } \ \alpha > 0$$

which implies the same exponential moment bound for τ.

Return times of a $M/G/\infty$ queue

We now state a general result on the tail behavior of the return time to zero $\mathcal{T}_1$ of an initially empty $M/G/\infty$ queue with a service time having exponential moments:
$\mathcal{T}_1$ then has basically the worse exponential moment between those of the inter-arrival time and of the service time.

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The result is based on the computation of the Laplace transform $\mathbb{E}(e^{-s\mathcal{T}_1})$ on the half-plane $\{s \in \mathbb{C} : \Re(s) > 0\}$ by Takács[10].
There the abscissa of convergence s_c is non-positive, but there is an apparent singularity on the pure imaginary axis.
We remove it using integration by parts and use Laplace transform theory and analytical continuation to prove that

$$\sigma_c \le -\gamma \quad \text{for some appropriate} \quad \gamma > 0\,.$$

[10] L. Takács (1956). "On a probability problem arising in the theory of counters". In: *Proc. Cambridge Philos. Soc.*

Theorem 7

Consider a $M/G/\infty$ queue with arrival rate $\lambda > 0$ and generic service time H such that $\mathbb{P}(H = 0) < 1$ and, for some $\gamma > 0$,

$$\mathbb{P}(H > t) \triangleq 1 - G(t) = O(\mathrm{e}^{-\gamma t}), \qquad t \geq 0 .$$

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The queue is started at 0. Let

- V_1 denote the arrival time of the first customer,
- $\mathcal{T}_1$ denote the subsequent time of return to 0 of the queue,
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Then the following holds.

1. If $\beta < \gamma$ then $\mathbb{E}(\mathrm{e}^{\beta B}) < \infty$. In particular $\mathbb{P}(B \geq t) = O(\mathrm{e}^{-\beta t})$.
2. If $\lambda < \gamma$ then $\mathbb{P}(\mathcal{T}_1 \geq t) = O(\mathrm{e}^{-\lambda t})$.
   If $\gamma \leq \lambda$ then $\mathbb{P}(\mathcal{T}_1 \geq t) = O(\mathrm{e}^{-\alpha t})$ for every $\alpha < \gamma$.

Elements of proof

We have $\mathcal{T}_1 = V_1 + B$ where V_1 and B are independent.
Takács[11] has proved that the Laplace transform of $\mathcal{T}_1$ satisfies

$$\mathbb{E}(\mathrm{e}^{-s\mathcal{T}_1}) = 1 - \frac{1}{\lambda + s} \frac{1}{\int_0^\infty \mathrm{e}^{-st-\lambda \int_0^t [1-G(u)]\,\mathrm{d}u}\,\mathrm{d}t}, \quad s \in \mathbb{C}, \ \mathfrak{R}(s) > 0 .$$

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Since the Laplace transform of V_1 is $\frac{\lambda}{\lambda+s}$, the Laplace transform of B satisfies

$$\mathbb{E}(\mathrm{e}^{-sB}) = \frac{\lambda + s}{\lambda} - \frac{1}{\lambda} \frac{1}{\int_0^\infty \mathrm{e}^{-st-\lambda \int_0^t [1-G(u)]\,\mathrm{d}u}\,\mathrm{d}t}, \quad s \in \mathbb{C},\ \Re(s) > 0,$$

and we study this formula.

[11] L. Takács (1956). "On a probability problem arising in the theory of counters". In: *Proc. Cambridge Philos. Soc.*

There is an apparent singularity in the r.h.s. at $\Re(s) = 0$ since

$$\lim_{s \to 0^+} \mathbb{E}(\mathrm{e}^{-sB}) = 1 \implies \lim_{s \to 0^+} \int_0^\infty \mathrm{e}^{-st-\lambda \int_0^t [1-G(u)]\,\mathrm{d}u}\,\mathrm{d}t = \infty\,.$$

It can be seen directly on the integral since

$$\int_0^\infty [1 - G(u)]\,\mathrm{d}u = \mathbb{E}[H] > 0\,.$$

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It can be seen directly on the integral since

$$\int_0^\infty [1-G(u)]\,du = \mathbb{E}[H] > 0\,.$$

We shall remove it by integration by parts and then prove that the abscissa of convergence σ_c of the Laplace transform $\mathbb{E}(e^{-sB})$ satisfies

$$\sigma_c \le -\gamma < 0\,.$$

We use integration by parts: on the half-line $\{s \in \mathbb{R} : s > 0\}$,

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-st-\lambda \int_0^t [1-G(u)]\,\mathrm{d}u}\,\mathrm{d}t \\
&= \left[\frac{\mathrm{e}^{-st}}{-s} \mathrm{e}^{-\lambda \int_0^t [1-G(u)]\,\mathrm{d}u} \right]_{t=0}^{\infty} \\
&\qquad - \int_0^\infty \frac{\mathrm{e}^{-st}}{-s} (-\lambda[1-G(t)])\,\mathrm{e}^{-\lambda \int_0^t [1-G(u)]\,\mathrm{d}u}\,\mathrm{d}t \\
&= \frac{1}{s} - \frac{\lambda}{s} \int_0^\infty [1-G(t)]\,\mathrm{e}^{-st-\lambda \int_0^t [1-G(u)]\,\mathrm{d}u}\,\mathrm{d}t\,.
\end{aligned}$$

Snce $1 - G(t) = O(\mathrm{e}^{-\gamma t})$ and

$$\lambda \int_0^\infty [1 - G(t)]\, \mathrm{e}^{-\lambda \int_0^t [1-G(u)]\,\mathrm{d}u}\, \mathrm{d}t = \left[-\mathrm{e}^{-\lambda \int_0^t [1-G(u)]\,\mathrm{d}u} \right]_{t=0}^\infty$$

$$= 1 - \mathrm{e}^{-\lambda \mathbb{E}(H)} < 1\,,$$

we define a constant $\theta < 0$ and an analytic function f by setting

$$\theta = \inf \left\{ s \leq 0 : \lambda \int_0^\infty [1 - G(t)]\, \mathrm{e}^{-st-\lambda \int_0^t [1-G(u)]\,\mathrm{d}u}\, \mathrm{d}t < 1 \right\} \vee (-\gamma)\,,$$

$$f(s) = \frac{\lambda + s}{\lambda} - \frac{s}{\lambda} \frac{1}{1 - \lambda \int_0^\infty [1 - G(t)]\, \mathrm{e}^{-st-\lambda \int_0^t [1-G(u)]\,\mathrm{d}u}\, \mathrm{d}t}\,, \quad \mathfrak{R}(s) > \theta\,.$$

The Laplace transform $\mathbb{E}(e^{-sB})$ has an abscissa of convergence $\sigma_c \leq 0$ and is analytic in the half-plane $\{s \in \mathbb{C} : \mathfrak{R}(s) > \sigma_c\}$, see Widder[12].

Both this Laplace transform and f are analytic in the domain $\{s \in \mathbb{C} : \mathfrak{R}(s) > \max(\theta, \sigma_c)\}$,

[12] David Vernon Widder (1941). *The Laplace Transform*.

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Both this Laplace transform and f are analytic in the domain $\{s \in \mathbb{C} : \Re(s) > \max(\theta, \sigma_c)\}$,

Since these two analytic functions coincide there on the half-line $\{s \in \mathbb{R} : s > 0\}$, they must coincide in the whole domain, see Rudin[13], so that

$$\mathbb{E}(\mathrm{e}^{-sB}) = f(s)\,, \qquad s \in \mathbb{C}\,,\ \Re(s) > \max(\theta, \sigma_c)\,.$$

Moreover, this Laplace transform must have an analytic singularity at $s = \sigma_c$, see Widder[10], and since f is analytic in $\{s \in \mathbb{C} : \Re(s) > \theta\}$ necessarily $\sigma_c \leq \theta$.

12 David Vernon Widder (1941). *The Laplace Transform*.

13 Walter Rudin (1987). *Real and complex analysis*. Third.

Since $\theta < 0$, by monotone convergence

$$\lim_{s\to\theta^+} f(s) = \frac{\lambda+\theta}{\lambda} - \frac{\theta}{\lambda}\frac{1}{1-\lambda\int_0^\infty [1-G(t)]\,\mathrm{e}^{-\theta t-\lambda\int_0^t [1-G(u)]\,\mathrm{d}u}\,\mathrm{d}t}$$
$$= \mathbb{E}(\mathrm{e}^{-\theta B}) \in [1,\infty]\,,$$

which implies that

$$\lambda\int_0^\infty [1-G(t)]\,\mathrm{e}^{-\theta t-\lambda\int_0^t [1-G(u)]\,\mathrm{d}u}\,\mathrm{d}t < 1,$$

and thus that $\theta = -\gamma$.
We conclude that

$$\sigma_c \le -\gamma\,.$$

Thus, if $\beta < \gamma$ then $\mathbb{E}(\mathrm{e}^{\beta B}) < \infty$.

The End

Thank you all for your attention

Sylvie, have a great 60th birthday conference