General criteria for the study of quasi-stationarity

Denis Villemonais
Collaboration with Nicolas Champagnat

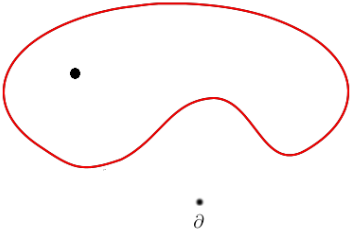
Université de Lorraine, Nancy, France

Populations: Interactions and Evolution, September 13, 2018

1. Absorbed processes

A generic example

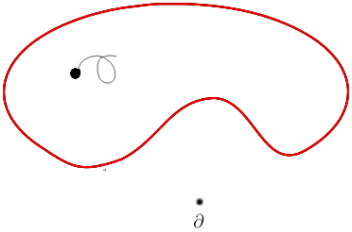
Process evolving stochastically in a domain $E \subset \mathbb{R}^2$, absorbed at the boundary



$\rightarrow$ $\partial \notin E$ unique absorbing point

A generic example

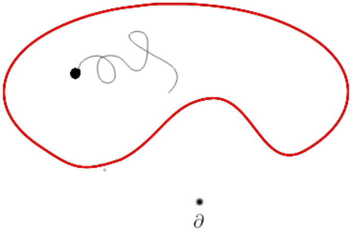
Process evolving stochastically in a domain $E \subset \mathbb{R}^2$, absorbed at the boundary



$\rightarrow$ $\partial \notin E$ unique absorbing point

A generic example

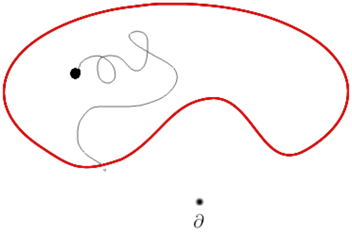
Process evolving stochastically in a domain $E \subset \mathbb{R}^2$, absorbed at the boundary



→ $\partial \notin E$ unique absorbing point

A generic example

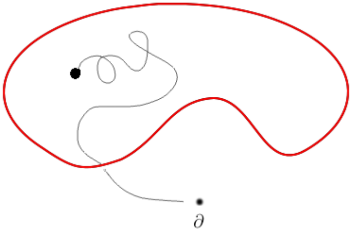
Process evolving stochastically in a domain $E \subset \mathbb{R}^2$, absorbed at the boundary



→ $\partial \notin E$ unique absorbing point

A generic example

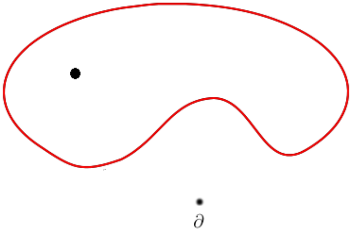
Process evolving stochastically in a domain $E \subset \mathbb{R}^2$, absorbed at the boundary



$\rightarrow$ $\partial \notin E$ unique absorbing point

A generic example

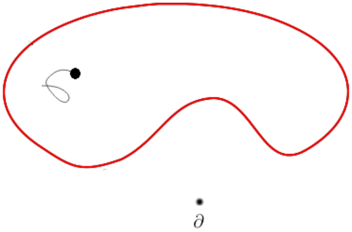
Process evolving stochastically in a domain $E \subset \mathbb{R}^2$, absorbed at the boundary



$\rightarrow$ $\partial \notin E$ unique absorbing point

A generic example

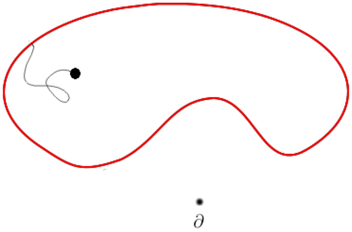
Process evolving stochastically in a domain $E \subset \mathbb{R}^2$, absorbed at the boundary



$\rightarrow$ $\partial \notin E$ unique absorbing point

A generic example

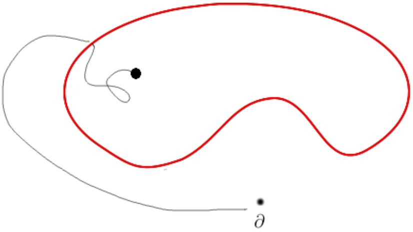
Process evolving stochastically in a domain $E \subset \mathbb{R}^2$, absorbed at the boundary



$\rightarrow$ $\partial \notin E$ unique absorbing point

A generic example

Process evolving stochastically in a domain $E \subset \mathbb{R}^2$, absorbed at the boundary



→ $\partial \notin E$ unique absorbing point

Formal defintion

Let $(X_t)_{t\in[0,+\infty[}$ evolving $E\cup\{\partial\}$, where $\partial\notin E$ is absorbing.

Denoting by $\tau_\partial=\inf\{t\geq 0,\ X_t=\partial\}$ the hitting time of ∂,

$$X_t=\partial,\ \forall t\geq\tau_\partial \text{ almost surely.}$$

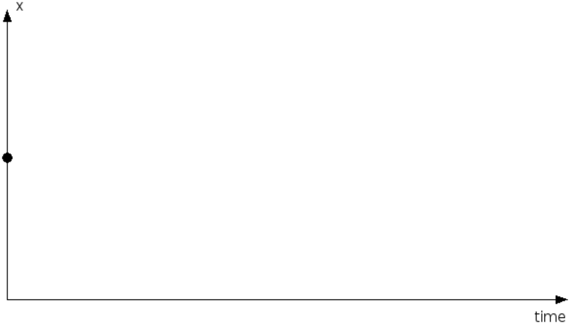
Formal defintion

Let $(X_t)_{t\in[0,+\infty[}$ evolving $E\cup\{\partial\}$, where $\partial\notin E$ is absorbing.

Denoting by $\tau_\partial=\inf\{t\geq 0,\ X_t=\partial\}$ the hitting time of ∂,

$$X_t=\partial,\ \forall\, t\geq\tau_\partial \text{ almost surely.}$$

Example : Process evolving in $E=[0,+\infty[$, with absorption at $\partial=0$



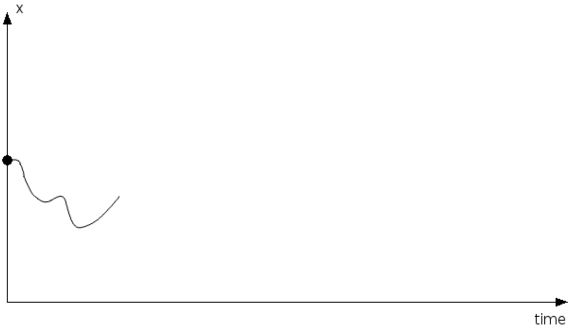
Formal defintion

Let $(X_t)_{t\in[0,+\infty[}$ evolving $E \cup \{\partial\}$, where $\partial \notin E$ is absorbing.

Denoting by $\tau_\partial = \inf\{t \geq 0,\ X_t = \partial\}$ the hitting time of ∂,

$$X_t = \partial,\ \forall t \geq \tau_\partial \text{ almost surely.}$$

Example : Process evolving in $E = [0, +\infty[$, with absorption at $\partial = 0$



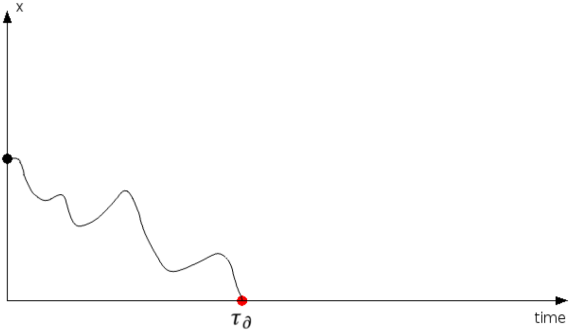
Formal defintion

Let $(X_t)_{t\in[0,+\infty[}$ evolving $E\cup\{\partial\}$, where $\partial\notin E$ is absorbing.

Denoting by $\tau_\partial=\inf\{t\geq 0,\ X_t=\partial\}$ the hitting time of ∂,

$$X_t=\partial,\ \forall t\geq\tau_\partial \text{ almost surely.}$$

Example : Process evolving in $E=[0,+\infty[$, with absorption at $\partial=0$



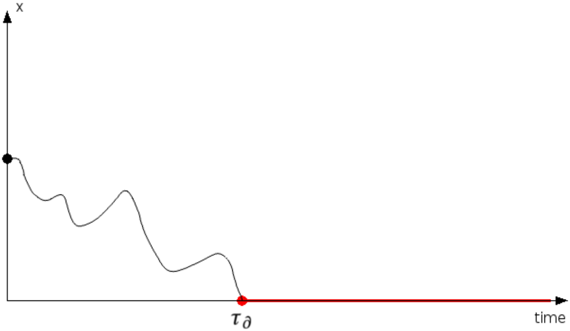
Formal defintion

Let $(X_t)_{t\in[0,+\infty[}$ evolving $E\cup\{\partial\}$, where $\partial\notin E$ is absorbing.

Denoting by $\tau_\partial=\inf\{t\geq 0,\ X_t=\partial\}$ the hitting time of ∂,

$$X_t=\partial,\ \forall t\geq\tau_\partial \text{ almost surely.}$$

Example : Process evolving in $E=[0,+\infty[$, with absorption at $\partial=0$



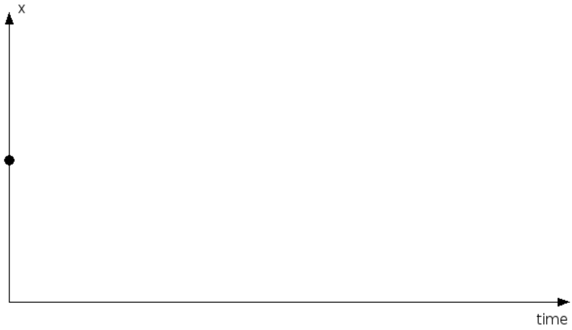
Formal defintion

Let $(X_t)_{t\in[0,+\infty[}$ evolving $E \cup \{\partial\}$, where $\partial \notin E$ is absorbing.

Denoting by $\tau_\partial = \inf\{t \geq 0,\ X_t = \partial\}$ the hitting time of ∂,

$$X_t = \partial,\ \forall t \geq \tau_\partial \text{ almost surely.}$$

Example : Process evolving in $E = [0, +\infty[$, with absorption at $\partial = 0$



Formal defintion

Let $(X_t)_{t\in[0,+\infty[}$ evolving $E\cup\{\partial\}$, where $\partial\notin E$ is absorbing.

Denoting by $\tau_\partial = \inf\{t\geq 0,\ X_t=\partial\}$ the hitting time of ∂,

$$X_t=\partial,\ \forall t\geq \tau_\partial \text{ almost surely.}$$

Example : Process evolving in $E=[0,+\infty[$, with absorption at $\partial = 0$

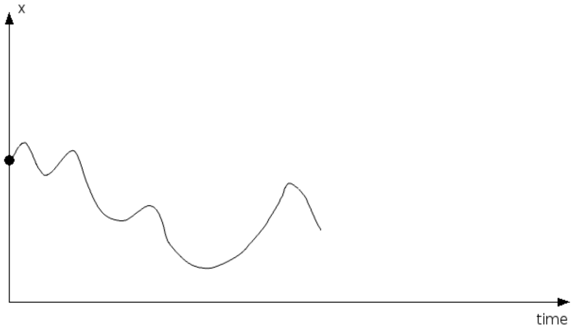
Formal defintion

Let $(X_t)_{t\in[0,+\infty[}$ evolving $E\cup\{\partial\}$, where $\partial\notin E$ is absorbing.

Denoting by $\tau_\partial=\inf\{t\geq 0,\ X_t=\partial\}$ the hitting time of ∂,

$$X_t=\partial,\ \forall t\geq\tau_\partial \text{ almost surely.}$$

Example : Process evolving in $E=[0,+\infty[$, with absorption at $\partial=0$



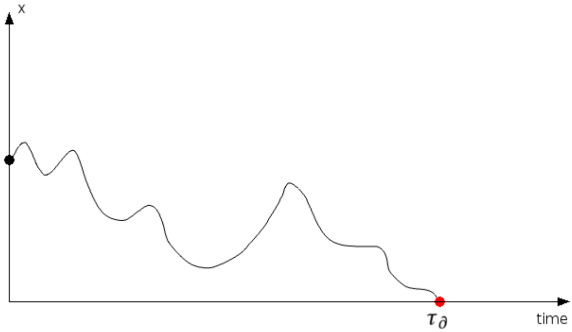
Formal defintion

Let $(X_t)_{t\in[0,+\infty[}$ evolving $E\cup\{\partial\}$, where $\partial\notin E$ is absorbing.

Denoting by $\tau_\partial=\inf\{t\geq 0,\ X_t=\partial\}$ the hitting time of ∂,

$$X_t=\partial,\ \forall\, t\geq\tau_\partial \text{ almost surely.}$$

Example : Process evolving in $E=[0,+\infty[$, with absorption at $\partial=0$



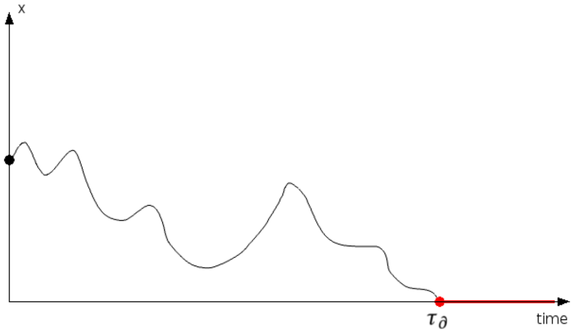
Formal defintion

Let $(X_t)_{t\in[0,+\infty[}$ evolving $E\cup\{\partial\}$, where $\partial\notin E$ is absorbing.

Denoting by $\tau_\partial=\inf\{t\geq 0,\ X_t=\partial\}$ the hitting time of ∂,

$$X_t=\partial,\ \forall t\geq\tau_\partial \text{ almost surely.}$$

Example : Process evolving in $E=[0,+\infty[$, with absorption at $\partial=0$

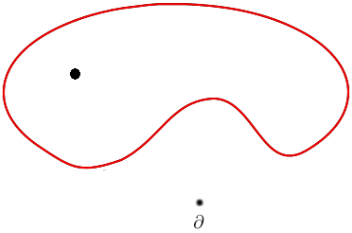


Limiting behaviour

In many interesting cases,

$$\mathbb{P}_{x_0}(X_t \in \cdot) \xrightarrow[t\to\infty]{} \delta_\partial, \ \forall x_0 \in E \cup \{\partial\}.$$

Question : What about the distribution of the process conditioned not to be absorbed, namely $\mathbb{P}_{x_0}(X_t \in \cdot \mid t < \tau_\partial)$?

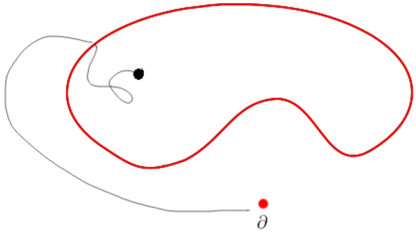


Limiting behaviour

In many interesting cases,

$$\mathbb{P}_{x_0}(X_t \in \cdot) \xrightarrow[t\to\infty]{} \delta_\partial, \ \forall x_0 \in E \cup \{\partial\}.$$

Question : What about the distribution of the process conditioned not to be absorbed, namely $\mathbb{P}_{x_0}(X_t \in \cdot \mid t < \tau_\partial)$?

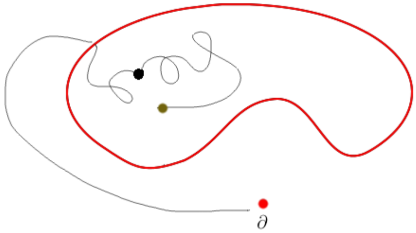


Limiting behaviour

In many interesting cases,

$$\mathbb{P}_{x_0}(X_t \in \cdot) \xrightarrow[t\to\infty]{} \delta_\partial, \ \forall x_0 \in E \cup \{\partial\}.$$

Question : What about the distribution of the process conditioned not to be absorbed, namely $\mathbb{P}_{x_0}(X_t \in \cdot \mid t < \tau_\partial)$?

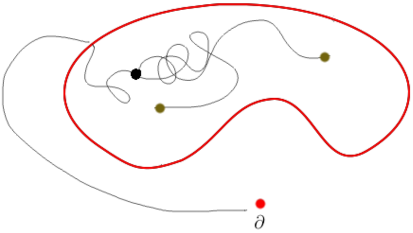


Limiting behaviour

In many interesting cases,

$$\mathbb{P}_{x_0}(X_t \in \cdot) \xrightarrow[t\to\infty]{} \delta_\partial, \ \forall x_0 \in E \cup \{\partial\}.$$

Question : What about the distribution of the process conditioned not to be absorbed, namely $\mathbb{P}_{x_0}(X_t \in \cdot \mid t < \tau_\partial)$?

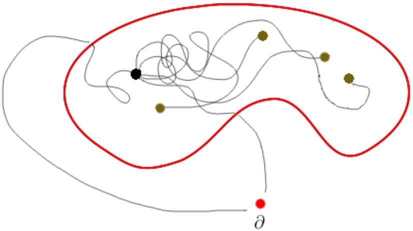


Limiting behaviour

In many interesting cases,

$$\mathbb{P}_{x_0}(X_t \in \cdot) \xrightarrow[t\to\infty]{} \delta_\partial, \ \forall x_0 \in E \cup \{\partial\}.$$

Question : What about the distribution of the process conditioned not to be absorbed, namely $\mathbb{P}_{x_0}(X_t \in \cdot \mid t < \tau_\partial)$?

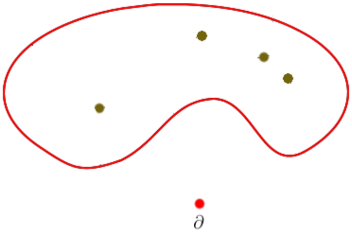


Limiting behaviour

In many interesting cases,

$$\mathbb{P}_{x_0}(X_t \in \cdot) \xrightarrow[t\to\infty]{} \delta_\partial, \ \forall x_0 \in E \cup \{\partial\}.$$

Question : What about the distribution of the process conditioned not to be absorbed, namely $\mathbb{P}_{x_0}(X_t \in \cdot \mid t < \tau_\partial)$?



Wright-Fisher model of genetical evolution

Discrete time/space. Let X, evolving in $E \cup \partial = \{0, 1, \ldots, N\}$, denotes the number of individuals with allele A in a population of size $N \geq 2$. Assume that the transition probabilities of X are given by

$$P(i,j) = \frac{N!}{j!(N-j)!} \left(\frac{i}{N}\right)^j \left(1 - \frac{i}{N}\right)^{N-j}.$$

$\partial = \{0, N\}$: allele A either disappears or invades the population.

Wright-Fisher model of genetical evolution

Discrete time/space. Let X, evolving in $E \cup \partial = \{0,1,\dots,N\}$, denotes the number of individuals with allele A in a population of size $N \geq 2$. Assume that the transition probabilities of X are given by

$$P(i,j) = \frac{N!}{j!(N-j)!} \left(\frac{i}{N}\right)^j \left(1 - \frac{i}{N}\right)^{N-j}.$$

$\partial = \{0,N\}$: allele A either disappears or invades the population.

Continuous time/space. Let X, evolving in $E \cup \partial = (0,1)$, denotes the proportion of individuals with allele A. Assume that

$$dX_t = \sqrt{X_t(1-X_t)}\, dB_t, \text{ with } B \text{ a Brownian motion.}$$

$\partial = \{0,1\}$: allele A either disappears or invades the population.

Penalized semi-groups

Consider a process X evolving in a state space E and add the following mechanism

- (killing) with rate $\kappa_k(X_t) \geq 0$, the particle is sent to a cemetary point $\partial \notin E$ and remains there,
- (branching) with rate $\kappa_b(X_t) \geq 0$, the process branches into two independent particles that follow the same dynamic as X (with killing and branching).

Penalized semi-groups

Consider a process X evolving in a state space E and add the following mechanism

- (killing) with rate $\kappa_k(X_t) \geq 0$, the particle is sent to a cemetary point $\partial \notin E$ and remains there,
- (branching) with rate $\kappa_b(X_t) \geq 0$, the process branches into two independent particles that follow the same dynamic as X (with killing and branching).

Then, denoting by $N_t \geq 0$ the number of particles at time t and by X^i the ith particle, one has

$$\mathbb{E}\left(\sum_{i=1}^{N_t} f(X_t^i)\right) = \mathbb{E}\left(e^{\int_0^t \kappa_b(X_s)-\kappa_k(X_s)\,ds} f(X_t)\right).$$

Penalized semi-groups

Consider a process X evolving in a state space E and add the following mechanism

- (killing) with rate $\kappa_k(X_t) \geq 0$, the particle is sent to a cemetary point $\partial \notin E$ and remains there,
- (branching) with rate $\kappa_b(X_t) \geq 0$, the process branches into two independent particles that follow the same dynamic as X (with killing and branching).

Then, denoting by $N_t \geq 0$ the number of particles at time t and by X^i the ith particle, one has

$$\mathbb{E}\left(\sum_{i=1}^{N_t} f(X_t^i)\right) = \mathbb{E}\left(e^{\int_0^t \kappa_b(X_s) - \kappa_k(X_s)\, ds} f(X_t)\right).$$

Multiplying this term by $e^{-t\|\kappa_b\|_\infty}$, one recovers the dynamic of a system with killing only, which fits into the settings of absorbed Markov processes.

2. Quasi-stationary distributions

Definition

A quasi-stationary distribution (QSD) is a probability measure α on E such that

$$\alpha = \lim_{t\to\infty} \mathbb{P}_\mu(X_t \in \cdot | t < \tau_\partial)$$

for some initial probability measure μ on E.

Proposition

A probability measure α is a QSD if and only if, for any $t \geq 0$,

$$\alpha = \mathbb{P}_\alpha(X_t \in .| t < \tau_\partial).$$

→ Surveys and book

- Méléard, V. 2012, Van Doorn, Pollett 2013
- Collet, Martínez, San Martín 2013

Proposition (Absorption rate admits a limit)

If$\alpha = \lim_{t\to\infty} \mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial)$, then there exists $\lambda_0 > 0$ such that

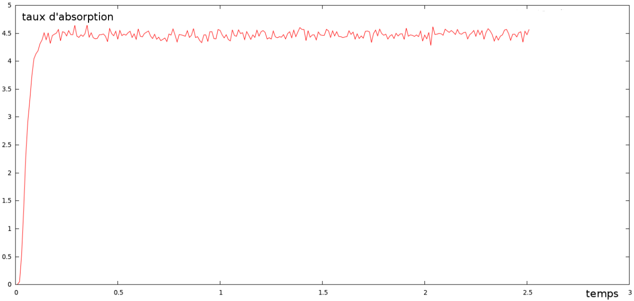
$$\text{absorbtion rate}(t) \stackrel{\text{def}}{=} \mathbb{P}^\partial_\mu(\tau_\partial \in]t, t+1] | \tau_\partial > t) \xrightarrow[t\to\infty]{} e^{-\lambda_0}.$$

Proposition (Absorption rate admits a limit)

If$\alpha = \lim_{t\to\infty} \mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial)$, then there exists $\lambda_0 > 0$ such that

$$\text{absorbtion rate}(t) \overset{\text{def}}{=} \mathbb{P}^\partial_\mu(\tau_\partial \in]t,t+1] | \tau_\partial > t) \xrightarrow[t\to\infty]{} e^{-\lambda_0}.$$

Brownian motion on $E =]0,1[$ absorbed at $\partial = \{0,1\}$.

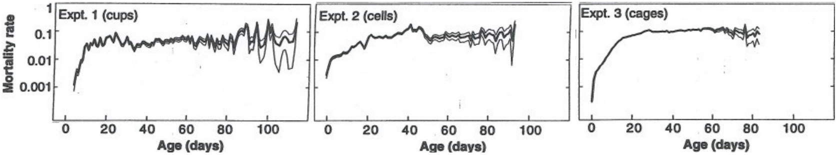


Proposition (Absorption rate admits a limit)

If $\alpha = \lim_{t\to\infty} \mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial)$, then there exists $\lambda_0 > 0$ such that

$$\text{absorbtion rate}(t) \stackrel{\text{def}}{=} \mathbb{P}^\partial_\mu(\tau_\partial \in]t, t+1] | \tau_\partial > t) \xrightarrow[t\to\infty]{} e^{-\lambda_0}.$$

Slowing of Mortality Rates at Older Ages In Large Medfly Cohorts (1992) Carey et al.



See also Steinsaltz & Wachter 2006

Définition

Let α be a QSD. The domain of attraction of α is the set of initial distributions μ such that

$$\lim_{t\to\infty} \mathbb{P}_\mu(X_t \in .|t < \tau_\partial) = \alpha.$$

Définition

Let α be a QSD. The domain of attraction of α is the set of initial distributions μ such that

$$\lim_{t\to\infty} \mathbb{P}_\mu(X_t \in .|t < \tau_\partial) = \alpha.$$

In the general case :

- **Existence** of a QSD is not true
- Existence does not imply **uniqueness** of a QSD
- Uniqueness does not imply **attraction of all initial distributions**
- Attraction of all initial distributions does not imply **uniform convergence**

Définition

Let α be a QSD. The domain of attraction of α is the set of initial distributions μ such that

$$\lim_{t\to\infty} \mathbb{P}_\mu(X_t \in .|t < \tau_\partial) = \alpha.$$

In the general case :

- **Existence** of a QSD is not true
- Existence does not imply **uniqueness** of a QSD
- Uniqueness does not imply **attraction of all initial distributions**
- Attraction of all initial distributions does not imply **uniform convergence**

Question : how to guarantee some or all of the above properties ?

3. Uniqueness of QSDs and exponential convergence

Probabilistic approach

Let X evolving in $E \cup \{\partial\}$ absorbed at ∂.

→ Assumption A1 (Doeblin condition)
There exists a probability measure ν and $c_1 > 0$ such that

$$\mathbb{P}_x(X_1 \in \cdot \mid 1 < \tau_\partial) \geq c_1 \nu(\cdot), \ \forall x \in E.$$

→ Assumption A2 (Harnack inequality)

$$\frac{\mathbb{P}_\nu(t < \tau_\partial)}{\mathbb{P}_x(t < \tau_\partial)} > c_2 > 0, \ \forall x \in E, t \geq 0.$$

Theorem (Champagnat, V. 2016)

A1 and A2 ⇔ there exists $C > 0$, $\gamma > 0$ and $\alpha \in \mathcal{M}_1(E)$ such that, for all $\mu \in \mathcal{M}_1(E)$,

$$\left\| \mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \alpha \right\|_{TV} \leq C e^{-\gamma t}.$$

Probabilistic approach

Let X evolving in $E \cup \{\partial\}$ absorbed at ∂.

- → Assumption A1 (Doeblin condition)
  There exists a probability measure ν and $c_1 > 0$ such that

$$\mathbb{P}_x(X_1 \in \cdot \,|\, 1 < \tau_\partial) \geq c_1 \nu(\cdot), \ \forall x \in E.$$

- → Assumption A2 (Harnack inequality)

$$\frac{\mathbb{P}_\nu(t < \tau_\partial)}{\mathbb{P}_x(t < \tau_\partial)} > c_2 > 0, \ \forall x \in E, t \geq 0.$$

Theorem (Champagnat, V. 2016)

A1 and A2 ⇔ there exists $C > 0$, $\gamma > 0$ and $\alpha \in \mathcal{M}_1(E)$ such that, for all $\mu \in \mathcal{M}_1(E)$,

$$\left\| \mathbb{P}_\mu(X_t \in \cdot | t < \tau_\partial) - \alpha \right\|_{TV} \leq C e^{-\gamma t}.$$

Probabilistic approach

Let X evolving in $E \cup \{\partial\}$ absorbed at ∂.

→ Assumption A1 (Doeblin condition)
There exists a probability measure ν and $c_1 > 0$ such that

$$\mathbb{P}_x(X_1 \in \cdot \,|\, 1 < \tau_\partial) \geq c_1 \nu(\cdot), \ \forall x \in E.$$

→ Assumption A2 (Harnack inequality)

$$\frac{\mathbb{P}_\nu(t < \tau_\partial)}{\mathbb{P}_x(t < \tau_\partial)} > c_2 > 0, \ \forall x \in E, t \geq 0.$$

Theorem (Champagnat, V. 2016)

A1 and A2 ⇔ there exists $C > 0$, $\gamma > 0$ and $\alpha \in \mathcal{M}_1(E)$ such that, for all $\mu \in \mathcal{M}_1(E)$,

$$\left\| \mathbb{P}_\mu(X_t \in \cdot | t < \tau_\partial) - \alpha \right\|_{TV} \leq C e^{-\gamma t}.$$

Comments and examples

- A1 and A2 have been used in several situations
  - general one dimensional diffusion processes
  - multi-dimensional diffusion processes (with K. Coulibaly-P)
  - birth and death processes with catastrophes
  - multi-dimensional birth and death processes
  - branching/dying Brownian motions
  - time-inhomogeneous processes
  - Benaïm, Cloez, Panloup 2016, Chazotte, Collet, Méléard 2017

Comments and examples

- A1 and A2 have been used in several situations
  - general one dimensional diffusion processes
  - multi-dimensional diffusion processes (with K. Coulibaly-P)
  - birth and death processes with catastrophes
  - multi-dimensional birth and death processes
  - branching/dying Brownian motions
  - time-inhomogeneous processes
  - Benaïm, Cloez, Panloup 2016, Chazotte, Collet, Méléard 2017
- A1 and A2 also imply several interesting properties
  - Spectral properties of the infinitesimal generator
  - Uniform convergence of $e^{\lambda_0 t}\mathbb{P}_x(t < \tau_\partial)$ toward an eigenfunction
  - Existence and exponential ergodicity of the Q-process

Comments and examples

- A1 and A2 have been used in several situations
  - general one dimensional diffusion processes
  - multi-dimensional diffusion processes (with K. Coulibaly-P)
  - birth and death processes with catastrophes
  - multi-dimensional birth and death processes
  - branching/dying Brownian motions
  - time-inhomogeneous processes
  - Benaïm, Cloez, Panloup 2016, Chazotte, Collet, Méléard 2017
- A1 and A2 also imply several interesting properties
  - Spectral properties of the infinitesimal generator
  - Uniform convergence of $e^{\lambda_0 t}\mathbb{P}_x(t < \tau_\partial)$ toward an eigenfunction
  - Existence and exponential ergodicity of the Q-process
- Intrinsec limitations
  - Uniform convergence and uniqueness of QSD
    $\Longrightarrow$ compact state spaces or entrance boundary at infinity
    or regularity of the boundaries
  - Not suited for the study of classical models (linear BD, Orstein-Uhlenbeck, AR-1, Galton-Watson, *etc...*)

5. A (far) more general criterion for the study of quasi-stationarity

A first model : perturbed dynamical systems

Let E be a measurable set of $\mathbb{R}^d$ with positive Lebesgue measure and let $\partial \notin E$. Assume that

$$X_{n+1} = \begin{cases} f(X_n) + \xi_n & \text{if } X_n \neq \partial \text{ and } f(X_n) + \xi_n \in E, \\ \partial & \text{otherwise,} \end{cases}$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is measurable and $(\xi_n)_{n \in \mathbb{N}}$ is an i.i.d. non-degenerate Gaussian sequence in $\mathbb{R}^d$.

Theorem (Champagnat, V. 2018)

If f is locally bounded such that

$$|x| - |f(x)| \xrightarrow[|x| \to +\infty]{} +\infty,$$

then there exists a quasi-stationary distribution attracting all initial distributions on E admitting an exponential moment.

A second model : diffusion processes

We consider a diffusion process X evolving in a bounded open domain $E \subset \mathbb{R}^d$ and absorbed at the boundary ∂E, solution to the SDE

$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt,\ X_0 \in E,$$

where σ and b are Hölder and uniformly elliptic.

A second model : diffusion processes

We consider a diffusion process X evolving in a bounded open domain $E \subset \mathbb{R}^d$ and absorbed at the boundary ∂E, solution to the SDE

$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt,\; X_0 \in E,$$

where σ and b are Hölder and uniformly elliptic.

Theorem (Champagnat, V. 2018)

There exists a unique quasi-stationary distribution.

A second model : diffusion processes

We consider a diffusion process X evolving in a bounded open domain $E \subset \mathbb{R}^d$ and absorbed at the boundary ∂E, solution to the SDE

$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt,\ X_0 \in E,$$

where σ and b are Hölder and uniformly elliptic.

Theorem (Champagnat, V. 2018)

There exists a unique quasi-stationary distribution.

As a corollary, we obtain the existence of a unique positive function η with C^2 regularity such that

$$-\lambda_0 \frac{\sigma\sigma^*}{2}\Delta\eta + b\cdot\nabla\eta$$

for some $\lambda_0 > 0$, *without any regularity condition on* ∂E.

Main ingredients

Common properties for these irreducible processes are that

- A1-A2 is satisfied *locally*
- there exist $\varphi_1 : E \to [1, +\infty)$ and $\varphi_2 : E \to [0,1]$ such that

  $$\mathbb{E}_x(\varphi_1(X_1)\mathbf{1}_{1<\tau_\partial}) \leq \theta_1 \varphi_1(x) + C_{st} \text{ and } \mathbb{E}_x(\varphi_2(X_1)\mathbf{1}_{1<\tau_\partial}) \geq \theta_2 \varphi_2(x)$$

  with $0 < \theta_1 < \theta_2$, φ_1 locally bounded and φ_2 locally positive.

$\longrightarrow$ these are the two main ingredients of our general criterion

Main ingredients

Common properties for these irreducible processes are that

- A1-A2 is satisfied *locally*
- there exist $\varphi_1 : E \to [1,+\infty)$ and $\varphi_2 : E \to [0,1]$ such that

  $$\mathbb{E}_x(\varphi_1(X_1)\mathbf{1}_{1<\tau_\partial}) \leq \theta_1\varphi_1(x) + C_{st} \text{ and } \mathbb{E}_x(\varphi_2(X_1)\mathbf{1}_{1<\tau_\partial}) \geq \theta_2\varphi_2(x)$$

  with $0 < \theta_1 < \theta_2$, φ_1 locally bounded and φ_2 locally positive.

$\longrightarrow$ these are the two main ingredients of our general criterion
Remark that, using Markov's property recursively, one obtains

$$\mathbb{E}(\varphi_1(X_n) \mid n < \tau_\partial)) \leq \frac{\mathbb{E}_x(\varphi_1(X_n)\mathbf{1}_{n<\tau_\partial})}{\mathbb{E}_x(\varphi_2(X_n)\mathbf{1}_{n<\tau_\partial})} \leq \left(\theta^n \frac{\varphi_1(x)}{\varphi_2(x)}\right) \vee C_\theta,\ \forall \theta \in (\theta_1/\theta_2, 1)$$

Main ingredients

Common properties for these irreducible processes are that

- A1-A2 is satisfied *locally*
- there exist $\varphi_1 : E \to [1,+\infty)$ and $\varphi_2 : E \to [0,1]$ such that

$$\mathbb{E}_x(\varphi_1(X_1)\mathbf{1}_{1<\tau_\partial}) \le \theta_1\varphi_1(x) + C_{st} \text{ and } \mathbb{E}_x(\varphi_2(X_1)\mathbf{1}_{1<\tau_\partial}) \ge \theta_2\varphi_2(x)$$

with $0 < \theta_1 < \theta_2$, φ_1 locally bounded and φ_2 locally positive.

$\longrightarrow$ these are the two main ingredients of our general criterion

Remark that, using Markov's property recursively, one obtains

$$\mathbb{E}(\varphi_1(X_n) \mid n < \tau_\partial)) \le \frac{\mathbb{E}_x(\varphi_1(X_n)\mathbf{1}_{n<\tau_\partial})}{\mathbb{E}_x(\varphi_2(X_n)\mathbf{1}_{n<\tau_\partial})} \le \left(\theta^n \frac{\varphi_1(x)}{\varphi_2(x)}\right) \vee C_\theta,\ \forall \theta \in (\theta_1/\theta_2, 1)$$

Hence, if $\varphi_1(x) \to +\infty$ at infinity, then the sequence $(\mathbb{P}_x(X_n \in \cdot \mid n < \tau_\partial))_{n\ge 0}$ is relatively compact and hence there exist limit distributions.

Application : convergence of a reinforced algorithm

Consider the process $(Y_t)_{t\geq 0}$ in $\mathbb{R}^d$ evolving as follows :

→ Y evolves following the SDE

$$dY_t = dB_t + b(X_t)\,dt,\ Y_0 \in \mathbb{R}^d \tag{1}$$

→ and, with rate $\kappa(Y_t) \geq 1$, the process jumps with respect to its empirical occupation measure $\frac{1}{t}\int_0^t \delta_{Y_s}\,\mathrm{d}s$.

Application : convergence of a reinforced algorithm

Consider the process $(Y_t)_{t\geq 0}$ in $\mathbb{R}^d$ evolving as follows :

→ Y evolves following the SDE

$$dY_t = dB_t + b(X_t)\,dt,\ Y_0 \in \mathbb{R}^d \tag{1}$$

→ and, with rate $\kappa(Y_t) \geq 1$, the process jumps with respect to its empirical occupation measure $\frac{1}{t}\int_0^t \delta_{Y_s}\,\mathrm{d}s$.

Theorem (Champagnat, V. 2018 ; Mailler, V. 2018)

Assume that $\limsup_{x\to+\infty} \frac{\langle b(x),x\rangle}{|x|} < -\frac{3}{2}\|\kappa\|_\infty^{1/2}$. Then

- the solution to (1) admits a unique QSD ν_{QSD} with an exponential moment of order $\|\kappa\|_\infty^{1/2}$

Application : convergence of a reinforced algorithm

Consider the process $(Y_t)_{t\geq 0}$ in $\mathbb{R}^d$ evolving as follows :

→ Y evolves following the SDE

$$dY_t = dB_t + b(X_t)\,dt,\ Y_0 \in \mathbb{R}^d \tag{1}$$

→ and, with rate $\kappa(Y_t) \geq 1$, the process jumps with respect to its empirical occupation measure $\frac{1}{t}\int_0^t \delta_{Y_s}\,\mathrm{d}s$.

Theorem (Champagnat, V. 2018 ; Mailler, V. 2018)

Assume that $\limsup_{x\to+\infty} \frac{\langle b(x),x\rangle}{|x|} < -\frac{3}{2}\|\kappa\|_\infty^{1/2}$. Then

- the solution to (1) admits a unique QSD ν_{QSD} with an exponential moment of order $\|\kappa\|_\infty^{1/2}$
- almost surely,

$$\frac{1}{t}\int_0^t \delta_{Y_s}\,\mathrm{d}s \xrightarrow[t\to+\infty]{weak} \nu_{QSD}.$$

Tiny non-exhaustive bibliography on QSDs

Seminal works

[1] Yaglom (1947) Certain limit theorems of the theory of branching random processes.

[2] Darroch, Seneta (1967) On quasi-stationary distributions in absorbing continuous-time ...

[3] Athrey, Ney (1972) Branching processes.

Book/surveys

[4] Collet, Martínez, San Martín (2013) Quasi-stationary distributions.

[5] van Doorn, Pollett (2013) Quasi-stationary distributions for discrete-state models.

[6] Méléard, V. (2012) Quasi-stationary distributions and population processes.

Diffusion processes

[7] Pinsky (1985) On the convergence of diffusion processes conditioned to remain in a bounded region ...

[8] Steinsaltz and Evans (2004) Markov mortality models : Implications of quasistationarity ...

[9] Cattiaux, Collet, Lambert, Martínez, Méléard and J. San Martín (2009) QSDs and ...

[10] Littin (2012) Uniqueness of quasistationary distributions and discrete spectra ...

[11] Kolb, Steinsaltz (2012) Quasilimiting behavior for one-dimensional diffusions with killing.

[12] Hening, Kolb (2016) QSDs for one-dimensional diffusions with singular boundary points.