Large Deviations in Epidemics Models

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joint work with B. Samegni–Kepgnou and B. Kouegou–Kamen

Poisson driven SDEs

- Consider the following SDE

$$Z_t^N = x_N + \frac{1}{N}\sum_{j=1}^k h_j P_j\left(N\int_0^t \beta_j(Z_s^N)ds\right),$$

where each of the d coordinates of Z_t^N takes its values in the set $\{k/N,\ k\in\mathbb{Z}_+\}$, provided the same is true with x_N and the coordinates of each vector h_j are either -1, 0 or 1. The P_j are i.i.d. standard Poisson processes. For $1\le j\le k$, β_j is locally Lipschitz from $\mathbb{R}_+^d$ into $\mathbb{R}_+$.

- The above SDE can be equivalently rewritten as

$$Z_t^N = x_N + \sum_{j=1}^k h_j \int_0^t \int_0^{N\beta_j(Z_{s-}^N)} Q_j(ds,du),$$

where the Q_j are i.i.d. Poisson random measures on $\mathbb{R}_+^2$, with mean measure $ds\,du$.

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Law of large numbers

- The following law of large numbers is essentially well–known (Kurtz'78), at least if the β_j are locally Lipschitz, which we do assume : $Z_t^N \to z_t$ a. s. locally uniformly in t as $N \to \infty$, where

$$\dot{z}_t = b(z_t), \ z_0 = \lim x_N,$$

if $b(z) = \sum_{j=1}^{k} \beta_j(z) h_j$.

- We are interested in situations where this ODE has a locally stable equilibrium, and we will exploit large deviations and the Wentzell–Freidlin theory, in order to describe the time it takes for the random perturbations inherent in the Z_t^N SDE to drive the system out of the basin of attraction of an equilibrium of the ODE.

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Examples of epidemics models 1

- The *SIS* model. If x_t denotes the proportion of infectious, and y_t the proportion of susceptible individuals in a population of constant size, where an individual is either infectious or susceptible (no immunity upon recovery). The law of large numbers ODE reads

$$\dot{x}_x = \lambda x_t(1 - x_t) - \gamma x_t,$$

where λ is the rate of infection and γ the recovery rate. If $R_0 = \lambda/\gamma > 1$, $x^* = (\lambda - \gamma)/\lambda$ is the stable endemic equilibrium.

- The *SIRS* model. In this model, when an individual looses its infection, he becomes "recovered" and immune, but he losses his immunity at rate ρ. The ODE model reads

$$\dot{x}_t = \lambda x_t y_t - \gamma x_t,$$
$$\dot{y}_t = -\lambda x_t y_t + \rho(1 - x_t - y_t).$$

Again if $R_0 > 1$, $z^* = \left(\frac{\rho}{\lambda}\frac{\lambda-\gamma}{\rho+\gamma}, \frac{\gamma}{\lambda}\right)$ is the stable endemic equilibrium, while the disease free equilibrium $(0, 1)$ is unstable.

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- The *SIR* model with demography. In this model, there is no loss of immunity, but there is an influx of susceptibles by births (or possibly immigration) at rate μ. We also assume that each individual, whether infectious, susceptible or removed, dies at rate μ. The ODE of this model reads

$$\dot{x}_t = \lambda x_t y_t - (\gamma + \mu) x_t,$$
$$\dot{y}_t = -\lambda x_t y_t + \mu - \mu y_t.$$

Here $R_0 = \lambda/(\gamma + \mu)$. If $R_0 > 1$, there is a stable endemic equilibrium, which is $(\frac{\mu}{\gamma+\mu}(1 - \frac{\gamma+\mu}{\lambda}), \frac{\gamma+\mu}{\lambda})$.

Examples of epidemics models 3

- The S_0IS_1 model of Safan, Heesterbeek and Dietz'06 (sort of intermediate between *SIS* and *SIRS*)

$$\dot{x}_t = -\mu x_t + \lambda(1 - x_t - y_t)x_t + r\lambda - \gamma x_t + r\lambda z_t x_t,$$
$$\dot{y}_t = -\mu y_t + \gamma x_t - r\lambda x_t y_t.$$

For certain values of the parameters, there is one locally stable endemic equilibrium, another one locally unstable, and a locally stable disease free equilibrium.

Large Deviations : the rate function

- We go back to the Poissonian SDE (x_N is the vector whose i–th coordinate reads $[Nx_i]/N$, is x is the starting point of the ODE)

$$Z_t^N = x_N + \frac{1}{N}\sum_{j=1}^{k} h_j P_j\left(N\int_0^t \beta_j(Z_s^N)ds\right).$$

- Let $\mathcal{AC}_{T,d}$ denote the set of absolutely continuous functions from $[0,T]$ into $\mathbb{R}^d$. If $\phi \in \mathcal{AC}_{T,d}$, we let $\mathcal{A}_\phi$ denote the (possibly empty) set of $c \in L^1([0,T];\mathbb{R}_+^k)$ with $\dot{\phi}_t = \sum_{j=1}^k c_j(t)h_j$. We define the rate function

$$I_T(\phi) := \begin{cases} \inf_{c\in\mathcal{A}_k(\phi)} I_T(\phi|c), & \text{if } \phi \in \mathcal{AC}_{T,d}; \\ \infty, & \text{otherwise,} \end{cases}$$

and with $g(a,b) = a\log(a/b) - a + b$,

$$I_T(\phi|c) = \int_0^T \sum_{j=1}^k g(c_j(t), \beta_j(\phi_t))dt$$

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The rate function 2

- Another formula for the rate function is

$$I_T(\phi) = \sup_{\theta \in C^1([0,T];\mathbb{R}^d)} \int_0^T \ell(\phi_t, \dot{\phi}_t, \theta_t) dt \,,$$

where

$$\ell(x, y, \theta) = \langle y, \theta \rangle - \sum_{j=1}^{k} \beta_j(x) \left(e^{\langle h_j, \theta \rangle} - 1 \right) .$$

- One can show that the two formulas define the same function, and that I_T is a good rate function on $D([0,T];\mathbb{R}^d)$. Note that $I_T(\phi) \geq 0$, and $I_T(\phi) = 0$ iff ϕ solves the ODE. I_T can be thought of as the cost of energy needed for diverting ϕ from solving the ODE.
- Large deviations. Proof of the lower bound based on a quasi–continuity result similar to Azencott'78, Priouret'82, and needs an assumption due to the fact that the β_j's may vanish. The upper bound for compact sets needs no specific assumption, while some restriction on the growth of the β_j's is needed for the exponential tightness.

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Quasi–continuity 1

- Let for $1 \le j \le k$ $Q_j^N(ds,du) = N^{-1}Q_j(ds,Ndu)$. The SDE can be rewritten as

$$Z_t^N = x_N + \sum_{j=1}^k h_j \int_0^t \int_0^{\beta_j(Z_{s-}^N)} Q_j^N(ds,du).$$

- Let $\phi \in \mathcal{AC}_{T,d}$ s.t. $K_\phi = \inf_{c\in\mathcal{A}_k(\phi)} \sum_{j=1}^k \int_0^T \frac{c_j(t)}{\beta_j(\phi_t)} dt < \infty$. The associate to ϕ the measures $\eta_j(ds,du)$ with the density

$$f_j(s,u) = \frac{c_j(s)}{\beta_j(\phi_s)} \mathbf{1}_{[0,\beta_j(\phi_s)]}(u) + \mathbf{1}_{(\beta_j(\phi_s),+\infty)}(u).$$

Then, with $x = \phi_0$, ϕ_t solves the ODE

$$\phi_t = x + \sum_{j=1}^k h_j \int_0^t \int_0^{\beta_j(\phi_s)} \eta_j(ds,du).$$

Define $A_{\phi,L} = \{(t,x),\ 0 \le t \le T,\ |x-\phi_t| \le L+1\}$, $\overline{\beta}(\phi,L) = \sup_{1\le j\le k} \sup_{(t,x)\in A_{\phi,L}} \beta_j(t,x)$.

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Quasi–continuity 2

We have the

Proposition

Let $T > 0$ be arbitrary. Given (ϕ, η) as above, such that in particular $K_\phi < \infty$, if $x_N = Z_0^N$, for any R, $L > 0$, there exists $\delta > 0$ (depending upon K_ϕ) and N_0 such that whenever $N \geq N_0$,

$$\mathbb{P}\left(\|Z^N - \phi\|_T > L,\ d_{T,\overline{\beta}}(Q^N, \eta) \leq \delta\right) \leq e^{-NR},$$

where

$$d_{T,\overline{\beta}}(\nu, \eta) = \sum_{j=1}^{k} \sup_{0\leq t\leq T,\, 0\leq u\leq\overline{\beta}} |\nu_j([0, t] \times [0, u]) - \eta_j([0, t] \times [0, u])|.$$

and $\overline{\beta} := \overline{\beta}(\phi, L)$.

Lower Bound

- Assume

(A.1) For any $\phi \in C([0,T];\mathbb{R}^d)$ such that $I_T(\phi) < \infty$ and any $\varepsilon > 0$, there exists ϕ^ε such that $\phi^\varepsilon_0 = \phi_0$, $K_{\phi^\varepsilon} < \infty$, $\|\phi - \phi^\varepsilon\|_T \le \varepsilon$ and $I_T(\phi^\varepsilon) \le I_T(\phi) + \varepsilon$.

- We have

Theorem

If the assumptions (A.1) is satisfied, then for any open subset $O \subset D([0,T];\mathbb{R}^d)$,

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(Z^N \in O\right) \ge -I_T(O).$$

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Upper Bound for compact sets

Without any further assumptions besides the locally Lipschitz property of te β_j', we can prove

Theorem

Let $T > 0$ be fixed. For any compact set $K \subset D([0,T];\mathbb{R}^d)$,

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(Z^N \in K\right) \leq -I_T(K).$$

The proof exploits the second formula for the rate function and the supermartingale property of certain exponentials.

Exponential tightness

- It remains to show that for any $\alpha > 0$, there exists a compact set $K_\alpha \subset\subset D([0,T];\mathbb{R}_+^d)$ such that

$$\limsup_{N\to\infty} N^{-1}\log\mathbb{P}(Z^N \notin K_\alpha) \le -\alpha.$$

- In order to establish that property, we need some growth restriction on the β_j's. The standard condition would be a sub–linear growth condition. This is not well adapted to the models we have in mind.
- We prove exponential tightness under the condition :

  (A.2) We assume that for all starting points $x_N \in \mathbb{Z}_+^d/N$, Z^N takes its values in $\mathbb{R}_+^d$ a.s., and moreover that there exists $C_\beta > 0$ such that for any j such that $\langle h_j, \mathbb{1}\rangle \neq 0$, $\beta_j(t,x) \le C_\beta(1+|x|),\ 0 \le t \le T,\ x \in \mathbb{R}^d$.

- A consequence of the above statements is that under both (A.1) and (A.2), the sequence $\{Z^N\}_{N\ge 1}$ satisfies the Large Deviation principle.

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Time of extinction of an endemic disease

- Let $T_0^N = \inf\{t > 0,\ Z_t^N \in \partial O\}$, where ∂O is the boundary of the basin of attraction of the endemic equil. z^*. In the three first examples, $T_0^N = \inf\{t > 0,\ Z_1^N(t) = 0\}$, where $Z_1^N(t) = N^{-1} \times \#$ of infectious individuals at time t.
- Define

$$V(z, z') = \inf_{T>0} \inf_{\phi_0 = z, \phi_T = z'} I_T(\phi),$$

$$\overline{V} = \inf_{z \in \partial O} V(z^*, z).$$

- We have

Theorem

Given $\eta > 0$, for all $z \in A$,

$$\lim_{N\to\infty} \mathbb{P}\left(\exp\{N(\overline{V} - \eta)\} < T_0^N < \exp\{N(\overline{V} + \eta)\}\right) = 1.$$

$\forall \eta > 0$, and N large enough, $\exp\{N(\overline{V} - \eta)\} \le \mathbb{E}(T_0^N) \le \exp\{N(\overline{V} + \eta)\}$.

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$\forall \eta > 0$, *and* N *large enough*, $\exp\{N(\overline{V} - \eta)\} \le \mathbb{E}(T_0^N) \le \exp\{N(\overline{V} + \eta)\}$.

The value of $\overline{V}$

- $\overline{V}$ is the value function of an optimal control problem. In the case of the *SIS* model, thanks to one dimensionality, one can exploit the Pontryagin maximum principle, and obtain the explicit formula : $\overline{V} = \log(\lambda/\gamma)$, so that $T_0^N \sim (\lambda/\gamma)^N$.
- In all other above examples, it looks like there is no explicit formula for $\overline{V}$. Of course, one can use numerical methods to compute their values for various sets of parameters.
- In the SIRS model, if we vary from $\lambda = 1.5$, $\gamma = 1$ and $\rho = 0.25$ to $\lambda = 20$, $\gamma = 15$, $\rho = 25$, then $\overline{V}$ varies from 8×10^{-3} to 4×10^{-5}.
- In the S_0IS_1 model with $\lambda = 3$, $\gamma = 5$, $\mu = 0.015$ and $r = 2$, we get $\overline{V} = 0.01$. This gives $e^{N\overline{V}}$ taking values from 2.7 to astronomical values. But with $\lambda = 28$, $\gamma = 10$, $\mu = 20$ and $r = 12$, $\overline{V} = 0.004$. For other values of the parameters, $e^{N\overline{V}} \simeq 1$ even for $N = 10^6$.

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- $\overline{V}$ is the value function of an optimal control problem. In the case of the *SIS* model, thanks to one dimensionality, one can exploit the Pontryagin maximum principle, and obtain the explicit formula : $\overline{V} = \log(\lambda/\gamma)$, so that $T_0^N \sim (\lambda/\gamma)^N$.
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Exit point 1

- Wentzell–Freidlin's theory tells us also what is the most probable part of the boundary which the process hits for large N. More precisely, suppose that there exists a unique point $\tilde{z} \in \partial O$ such that $V(z^*, \tilde{z}) = \overline{V}$, so that $V(z^*, z) > \overline{V}$, for all $z \in \partial O \backslash \{\tilde{z}\}$, and in the non–compact case there exists a compact $K \ni \tilde{z}$ and $c > 0$ such that $V(z^*, z) \geq \overline{V} + c$ for $z \in \partial O \cap K^c$. Under those assumptions, for any $\delta > 0$,
$$\lim_{N\to\infty} \mathbb{P}(\|Z^N_{T^N_0} - \tilde{z}\| < \delta) = 1.$$
- In the four examples above, there is one special point on ∂O, let us call it $\tilde{z}$, which is the stable equilibrium of the ODE when restricted to the boundary. It is obvious that $V(z^*, \tilde{z}) = \overline{V}$, but it is not obvious that this is the unique minimum.

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Exit point 2

- However, in all four above examples, one can show that $\tilde{z}$ is the unique minimum. The sketch of the argument is as follows.
- If there exists an optimal trajectory going from z^* to some $z \in \partial O \backslash \{\tilde{z}\}$ in finite time, then concatenating that trajectory with the solution of the ODE starting from z, we would have an optimal trajectory for the control problem with the same cost functional, but with the constraints $\phi_0 = z^*$, $\phi_T = \tilde{z}$.
- From the Pontryagin maximum principle, there would exist a continuous adjoint state, which is zero along the solution of the ODE, so would be zero when reaching z. But in each example one can show that this is impossible.
- So an optimal trajectory could reach the boundary at $z \in \partial O \backslash \{\tilde{z}\}$ only in infinite time. But in infinite time, an optimal trajectory reaches the boundary necessarily at $\tilde{z}$ (this uses again the Pontryagin maximum principle).

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Bibliography

- T. Britton and E. Pardoux, Stochastic epidemics in a homogeneous community, submitted.
- M.I. Freidlin and A.D. Wentzell, *Random perturbations of dynamical systems*, Springer Verlag 1984.
- P. Kratz and E. Pardoux, Large deviations for epidemiological models, *Séminaire de Probabilités* XLIX, C. Donati-Martin, A. Lejay, A. Rouault eds., Lecture Notes in Math. **2215**, pp. 221–327, 2018.
- T.G. Kurtz, Strong approximation theorems for density dependent Markov chains. *Stoch. Proc. and App.*, **6**, 223–240, 1978.
- E. Pardoux, B. Samegni–Kepgnou, Large Deviation Principle for Poisson driven SDEs in Epidemic models, *J. Applied Probab.* **54**, 905–920, 2017.
- E. Pardoux, B. Samegni–Kepgnou, Problem of exit through a characteristic boundary for a Poisson driven SDE, submitted.
- A. Shwartz and A. Weiss, Large deviations with diminishing rates, *Mathematics of Operations Research* **30** 281–310, 2005.

HAPPY BIRTHDAY SYLVIE !