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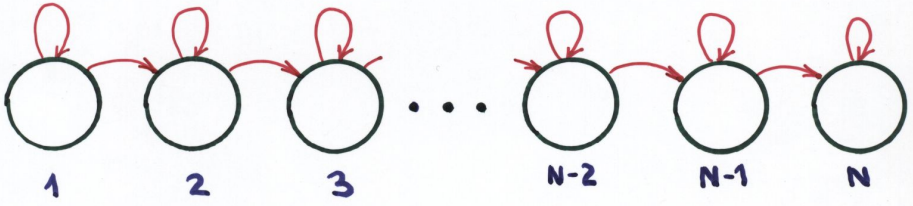
Coexistence of types in decomposable processes

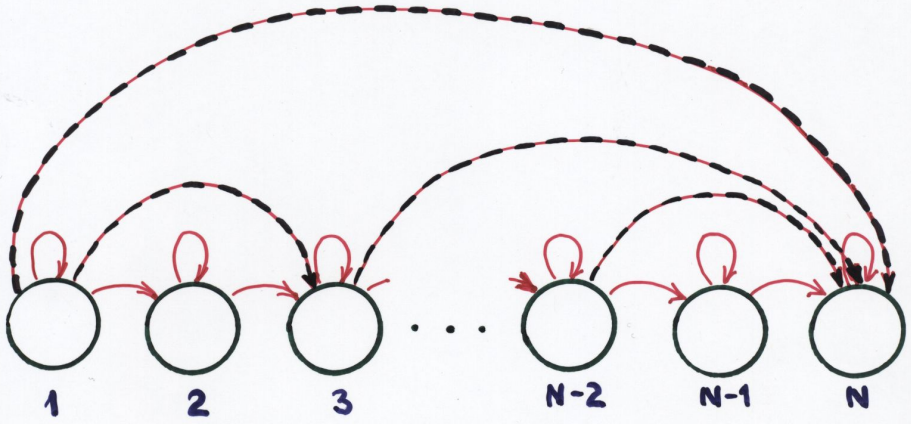
Vladimir Vatutin (Steklov Mathematical Institute)

Paris, September 10th, 2018

Island model



Island model



The number assigned to an island $=$ the type of particles located on the island.

The population vector at time n is

$$\mathbf{Z}(n) = (Z_1(n), ..., Z_N(n)).$$

η_{ij} – the numbers of daughters of type j a mother of type $i \in \{1, 2, ..., N\}$ with $\eta_{ij} = 0$ if $i > j$. Thus,

$$(0, ..., 0, \eta_{ii}, \eta_{i,i+1}, ..., \eta_{iN})$$

is the offspring vector of a type i particle with (generally) **dependent** components.

Hypothesis A: (finite variance): the $N-$type decomposable process is **strongly** critical:

$$m_{ii} = \mathbf{E}[\eta_{ii}] = 1, \quad i = 1, 2, ..., N,$$

and

$$b_i = \frac{1}{2} Var[\eta_{ii}] \in (0, \infty), i = 1, 2, ..., N,$$

$$m_{i,i+1} = \mathbf{E}[\eta_{i,i+1}] \in (0, \infty), \ i = 1, 2, ..., N-1.$$

The mean matrix $\mathbf{M} = (m_{ij})_{i,j=1}^{N}$ has the form

$$\mathbf{M} = \begin{pmatrix} 1 & m_{12} & \dots & \dots & m_{1N} \\ 0 & 1 & m_{23} & \dots & m_{2N} \\ 0 & 0 & 1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & m_{N-1,N} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Theorem

(Foster,Ney, 1978) *Let* $\{\mathbf{Z}(n), n = 0, 1, ..\}$ *be a strongly critical decomposable multitype branching process satisfying* ***Hypothesis A****. Then, as* $n \to \infty$

$$Q_n^{(1,N)} = \mathbf{P}(Z_1(n) + ... + Z_N(n) > 0) \sim c_{1N} n^{-1/2^{N-1}}.$$

For instance

$$Q_n^{(1,1)} \sim \frac{c_{11}}{n}, \qquad Q_n^{(1,2)} \sim \frac{c_{12}}{\sqrt{n}}, \qquad Q_n^{(1,3)} \sim \frac{c_{13}}{n^{1/4}}, \qquad Q_n^{(1,4)} \sim \frac{c_{14}}{n^{1/8}}$$

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Hence we have

$$\mathbf{P}(Z_1(n) + ... + Z_N(n) > 0) \sim \mathbf{P}(Z_N(n) > 0; Z_1(n) + ... + Z_{N-1}(n) = 0).$$

Islands

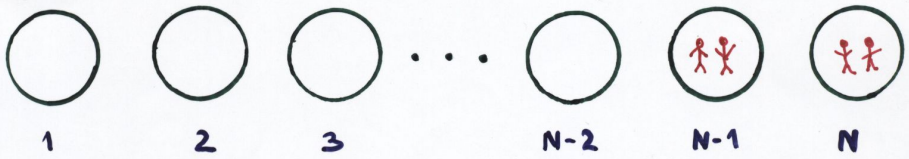
$t = n$

Islands

1) $t = n$

2) $n^{1/2} \ll t \leq n$

3) $t \asymp n^{1/2}$



Islands

4) $n^{1/4} \ll t \ll n^{1/2}$

Islands

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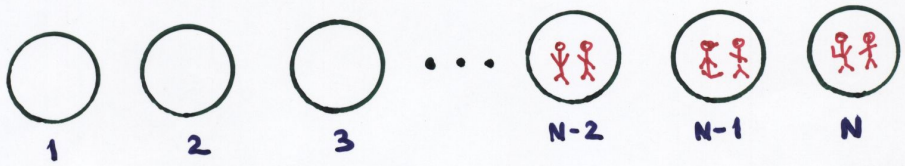
5) $t \asymp n^{1/4}$



Islands

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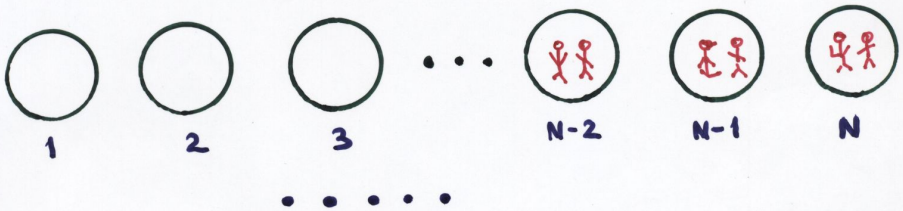


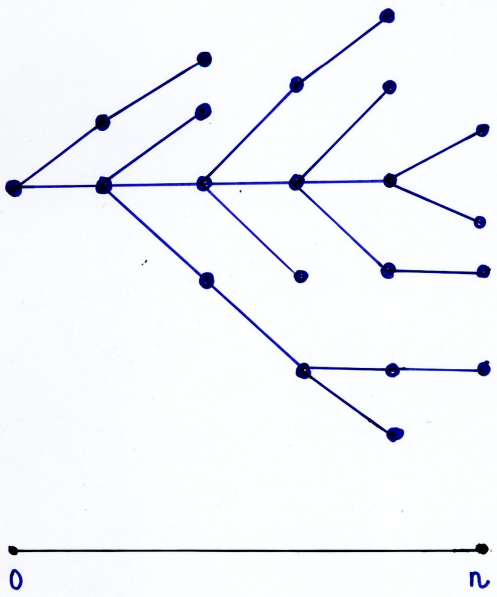
. . . . .

6) $t \asymp n^{1/2^{N-1}}$

Islands

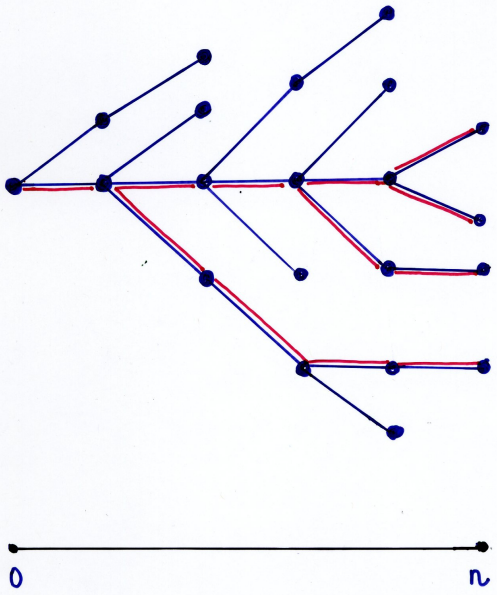
$t = n$

Genealogical tree for an ordinary process



Full! Tree

Genealogical tree for the respective reduced process



Full! Tree

Reduced ! Tree

REDUCED PROCESSES

Let $Z_i\,(m,n)$ be the number of type i particles existing in the process at moment $m < n$ and having nonempty number of descendants at moment n and let

$$\mathbf{Z}(m,n) = (Z_1(m,n), ..., Z_N(m,n)).$$

Problem to be considered:

• The distribution of $\left\{\mathbf{Z}(m,n)|\mathbf{Z}(n) \neq \mathbf{0}\right\}$ when $m,n \to \infty$ in a proper way.

The limiting process for the macroscopic **picture** as $n \to \infty$:

$$\Big(\mathbf{Z}(n^v \log n, n), 0 \le v < 1 \Big| \mathbf{Z}(n) \ne \mathbf{0}\Big) \rightleftarrows \Big(\mathbf{R}(v), 0 \le v < 1\Big),$$

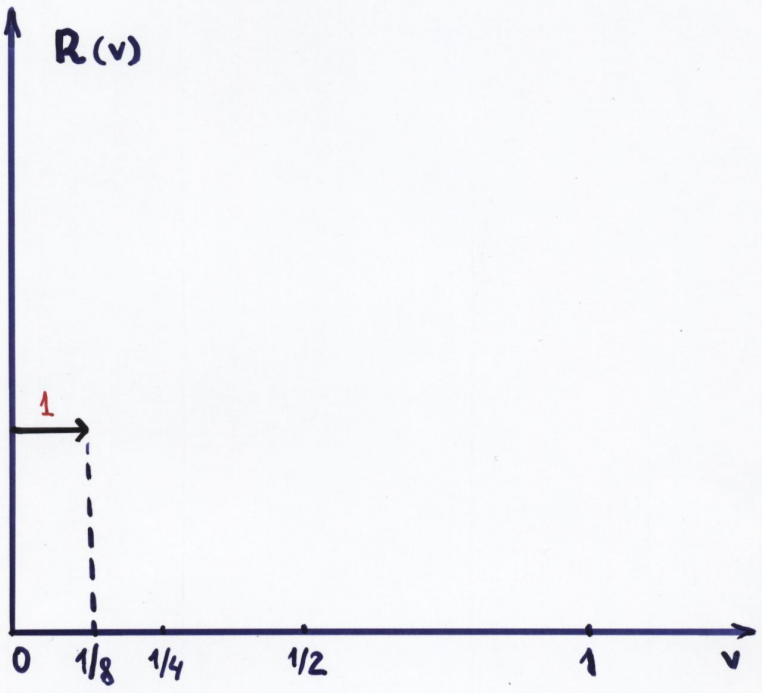
where

$$\mathbf{R}(v) = (R_1(v), ..., R_N(v)), 0 \le v < 1,$$

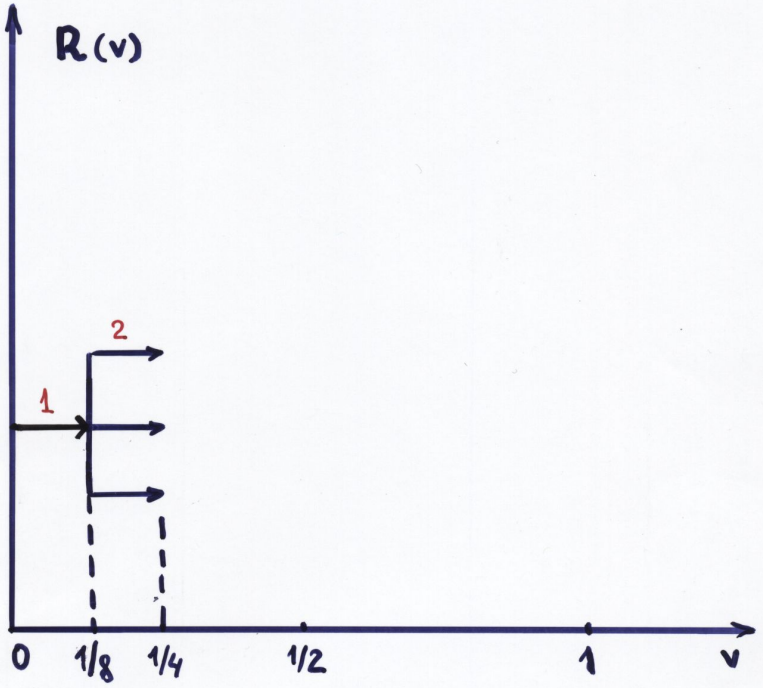
is an $N-$type decomposable branching process,

$R_i(v)$ is the number of type i individuals in the population at moment v.

$N = 4, \quad 1/2^{N-1} = 1/8$



$N = 4, \quad 1/2^{N-1} = 1/8, \qquad f(s) = 1 - \sqrt{1-s},\ f'(1) = \infty$



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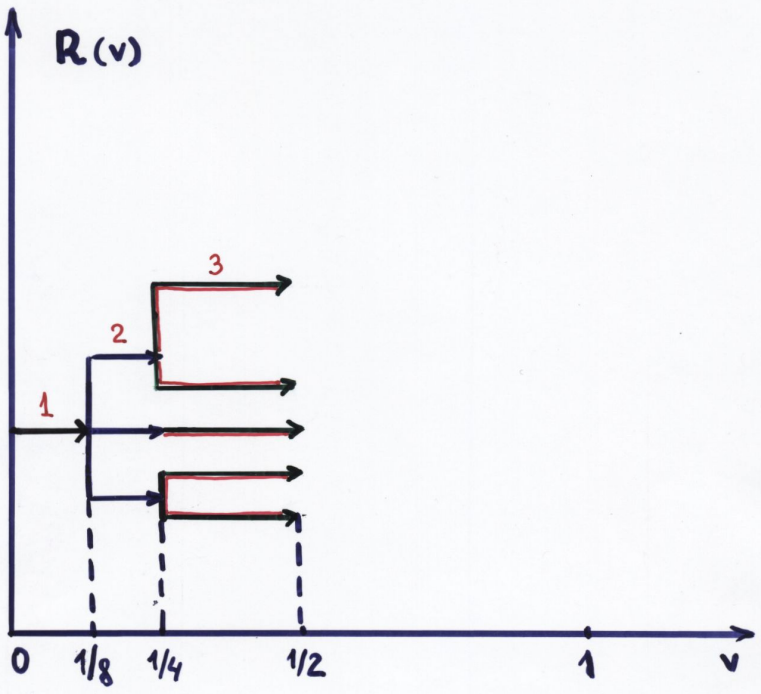

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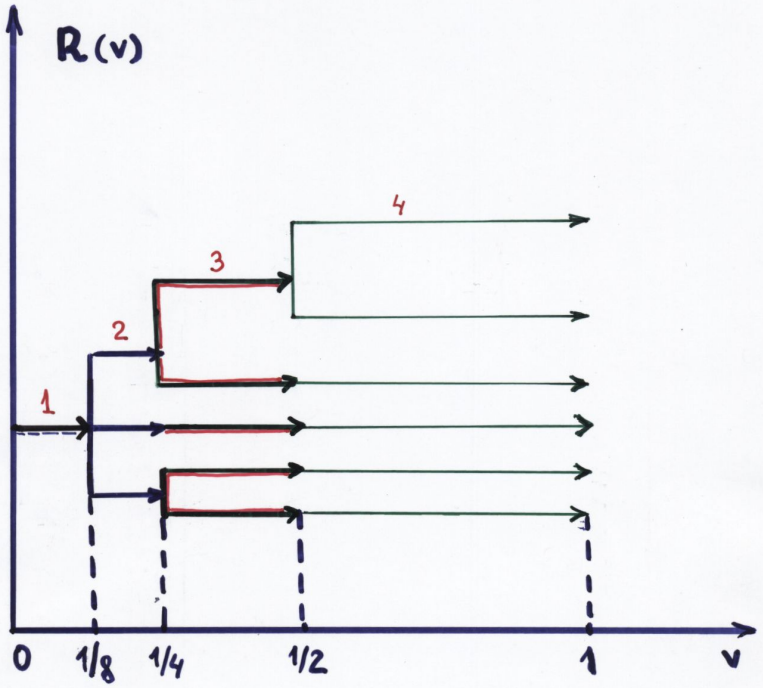


Theorem

As $n \to \infty$ *the finite-dimensional distributions of the process*

$$\{(\mathbf{Z}(n^v \log n, n), 0 \le v < 1) | \mathbf{Z}(n) \neq \mathbf{0}\}$$

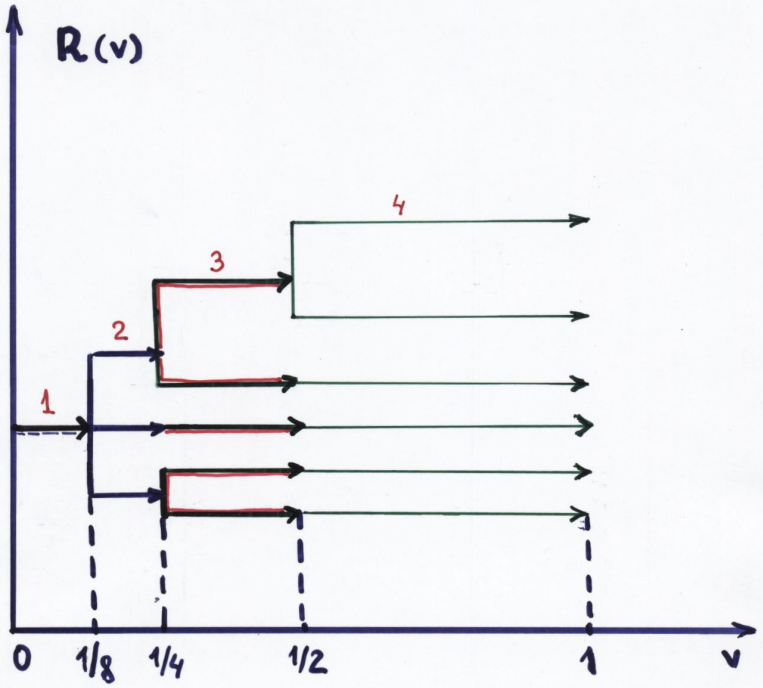
converge to the finite-dimensional distributions of $\{\mathbf{R}(v), 0 \le v < 1\}$.

At each (macroscopic) **moment in the past the ancestors are located** on a single island **only!**

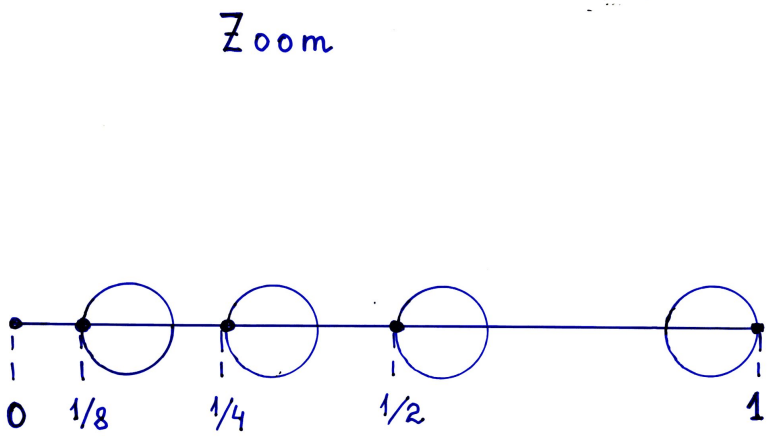
or

At each (macroscopic) **moment in the past the ancestors are** of a single type **only!**

$N = 4, \quad 1/2^{N-1} = 1/8,$?? Phase transition??



Zoom around the points of phase transition



The limiting process for the microscopic **picture**:

$$\left\{\mathbf{Z}\left(\left(y+\frac{1}{\log n}\right)n^{1/2^{N-i}},n\right),0\le y<\infty\right\}\rightleftarrows\left\{\mathbf{U}_i(y),0\le y<\infty\right\}$$

where

$$\mathbf{U}_i(y)=(U_{i1}(y),...,U_{iN}(y)),\ 0\le y<\infty,\ i=1,2,...,N-1,$$

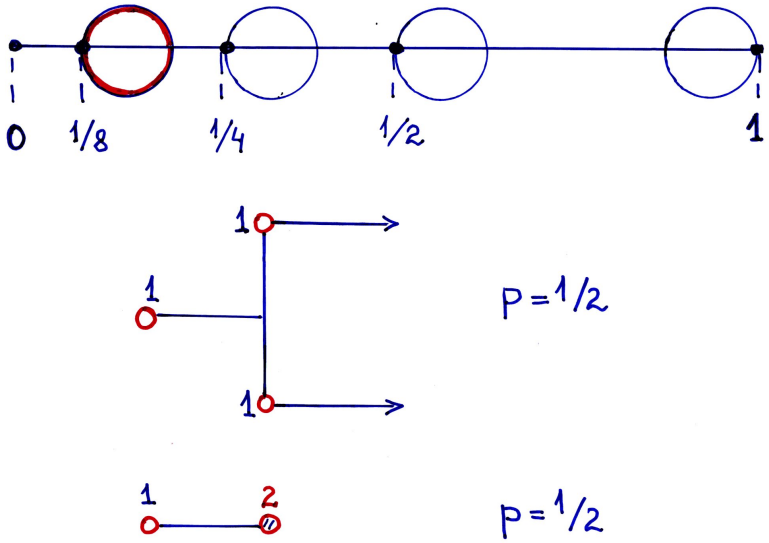
is a time-continuous Markov branching process with

$$U_{ij}(y)\equiv 0,\ 0\le y<\infty,\ j\ne i,i+1,$$

Zoom around the points of phase transition: $N = 4, 1/2^{N-1} = 1/8$

$$\left\{\mathbf{Z}\left(\left(y + \frac{1}{\log n}\right)n^{1/8}, n\right), 0 \le y < \infty\right\} \rightleftarrows \left\{\mathbf{U}_1(y), 0 \le y < \infty\right\}$$

Zoom



The pair

$$(U_{ii}(y), U_{i,i+1}(y)), 0 \leq y < \infty,$$

constitutes a **two-type** time-homogeneous Markov branching process with particles of types i and $i+1$.

$$\mathbf{E}\left[s_i^{U_{ii}(0)}\right] = 1 - (1 - s_i)^{1/2^{i-1}}, \qquad U_{i,i+1}(0) = 0.$$

The life-length distribution of type i particles is exponential.

Dying each particle of type i produces either **two children** of type i or **one child** of type $i+1$ (each option with probability $1/2$.)

Particles of type $i+1$ of $\mathbf{U}_i(\cdot)$ are **immortal** at the microscopic scale **and produce no children.**

At each moment $y > 0$ the ancestors are located on two islands only.

Theorem

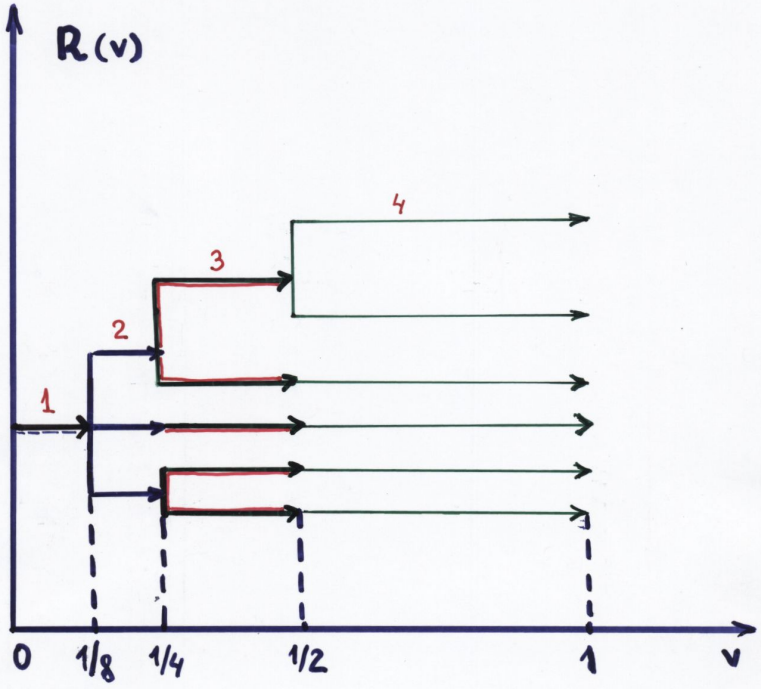
As $n \to \infty$ for $i = 1, 2, ..., N-1$

$$\left\{ (\mathbf{Z}\left(\left(y + \frac{1}{\log n}\right) n^{1/2^{N-i}}, n\right), 0 \le y < \infty) \middle| \, \mathbf{Z}(n) \neq \mathbf{0} \right\}$$

$$\Longrightarrow \{\mathbf{U}_i(y), 0 \le y < \infty\}.$$

J.H.Gillespie, The Causes of Molecular Evolution:
There are mathematical models of evolution in the framework of which we must conclude that evolution is episodic, with bursts of substitutions (of lengths having orders $n^{1/2^{N-1}}, n^{1/2^{N-2}}, ..., n^{1/2}$ in our case) separated by periods of quiescence.

N=4



The limiting process for the microscopic picture when time is of order n:

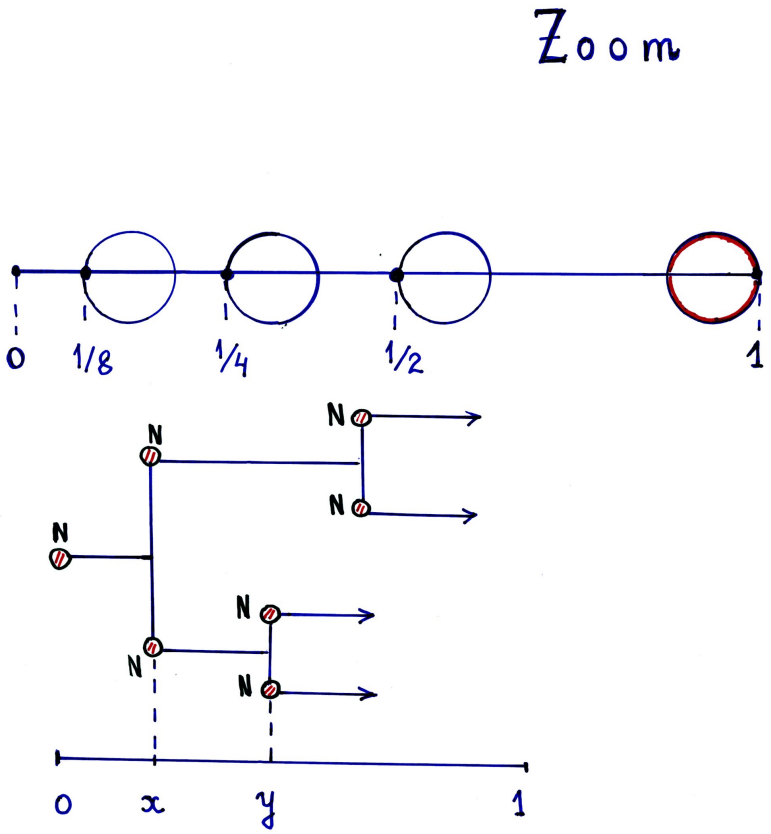
$$\left\{\mathbf{Z}\left(\left(x+\frac{1}{\log n}\right)n, n\right), 0 \le x < 1\right\} \Longrightarrow \left\{\mathbf{U}_N(x), 0 \le x < 1\right\}.$$

where

$$\mathbf{U}_N(x) = (0, ..., 0, U_{NN}(x)),\ 0 \le x < 1,$$

is a time-continuous Markov branching process with first $N-1$ components equal to zero.

Zoom around the points of phase transition



$U_{NN}(\cdot)$ is a single-type inhomogeneous Markov branching process with

$$\mathbf{E}\left[s_N^{U_{NN}(0)}\right] = 1 - (1 - s_N)^{1/2^{N-1}}.$$

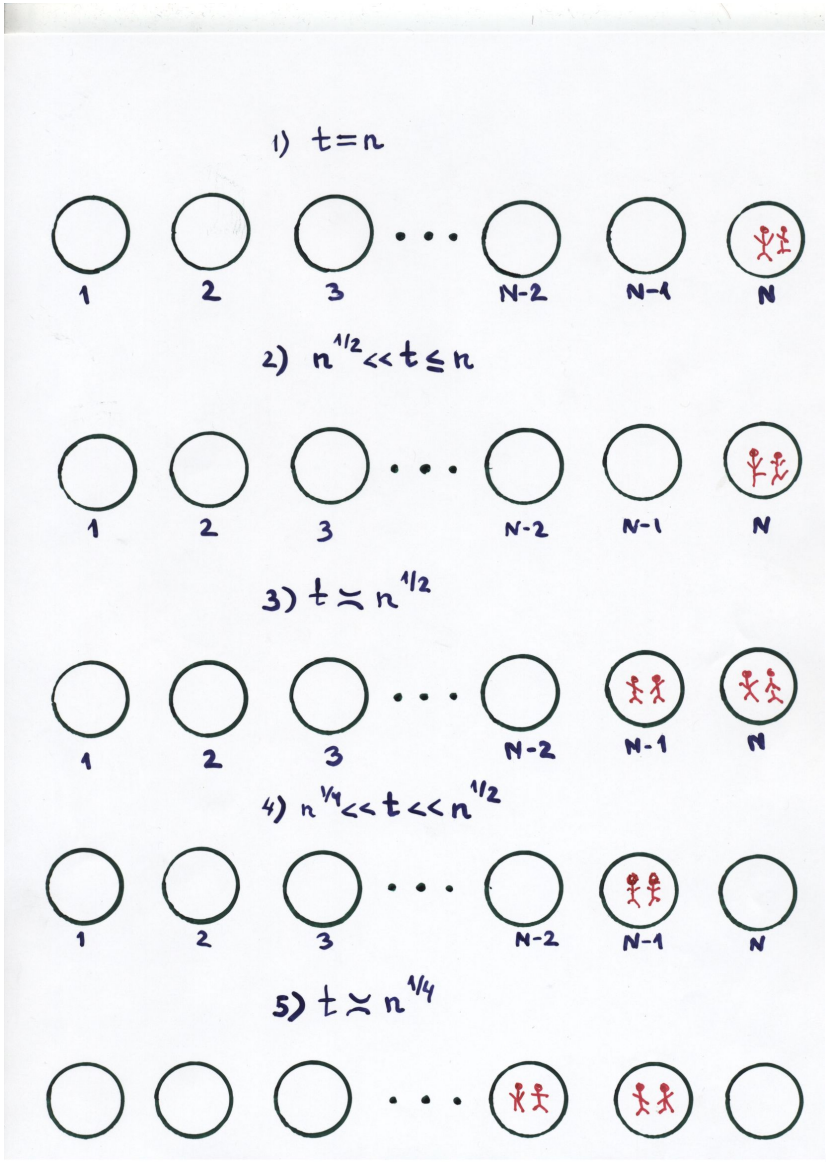
The life-length of each of $U_{NN}(0)$ type N initial particles is uniformly distributed on the interval $[0, 1]$.

Dying such a particle produces exactly **two children** of type N.

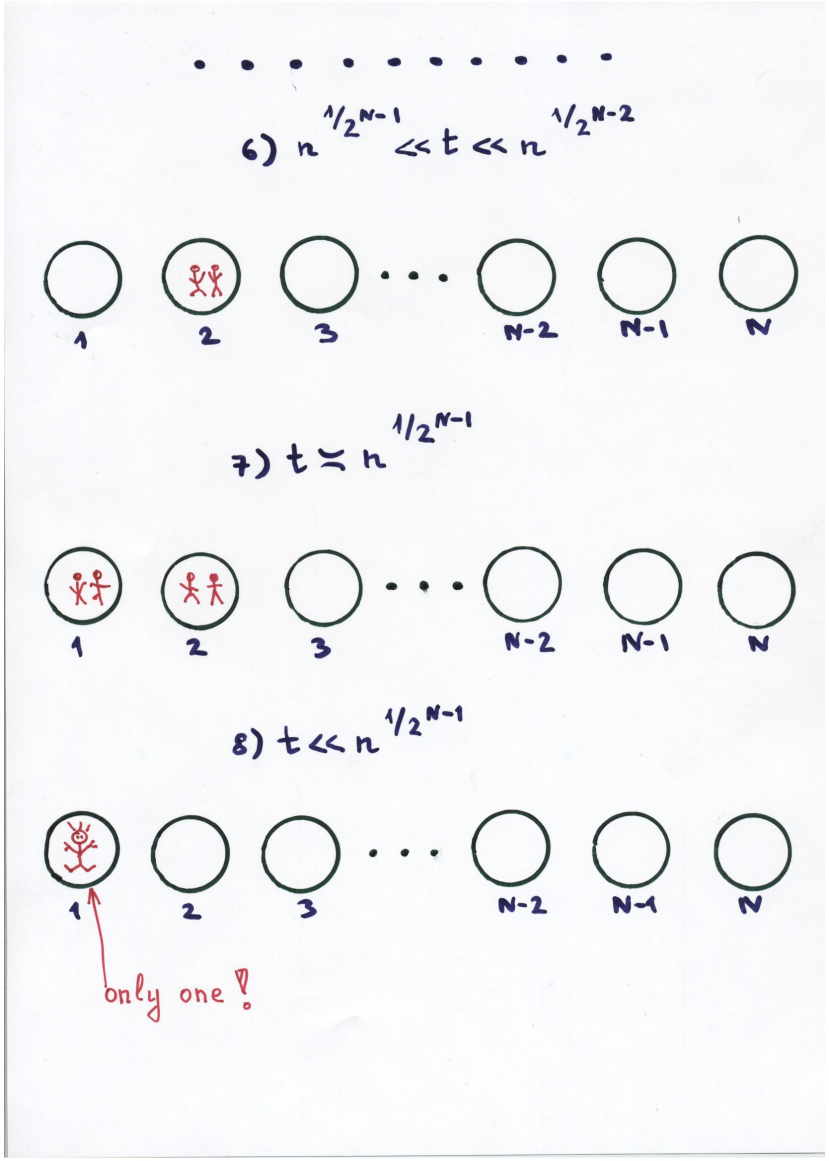
If the death moment of the parent particle is x then the life length of each of its offspring has the uniform distribution on the interval $[x, 1]$. Dying each particle of the process produces exactly two individuals of type N and so on...

The case $N = 1$ has been considered (independently) by **Zubkov (1975) and Fleischmann, Siegmund-Schultze (1977).**

Reduced process



Reduced process



Is this the only possible scenario????

NO!!

A **two-type** decomposable Galton-Watson branching process where the offspring size of particles may have infinite variance **(Smadi+V.)**

Hypothesis B (infinite variance) As $s_1, s_2 \uparrow 1$:

$$F_{12}(s_1, s_2) = s_1 + (1 - s_1)^{1+\alpha_1} L_1 (1 - s_1) - (A_{12} - \rho (s_1, s_2)) (1 - s_2),$$

and

$$F_2(s_2) = s_2 + (1 - s_2)^{1+\alpha_2} L_2 (1 - s_2),$$

where $0 < \alpha_1, \alpha_2 \leq 1$, $L_1(x)$ and $L_2(x)$ are slowly varying functions as $x \downarrow 0$,

$$A_{12} = \mathbf{E} [Z_2(1) | \mathbf{Z}(0) = \mathbf{e}_1] \in (0, \infty),$$

and $\rho (s_1, s_2) \to 0$ as $s_1 s_2 \uparrow 1$.

Hypothesis B (infinite variance) As $s_1, s_2 \uparrow 1$:

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$$A_{12} = \mathbf{E} [Z_2(1) | \mathbf{Z}(0) = \mathbf{e}_1] \in (0, \infty),$$

and $\rho (s_1, s_2) \to 0$ as $s_1 s_2 \uparrow 1$.

It is known **(Slack, 1968)** that in this case

$$\mathbf{P} (Z_i(n) > 0 | \mathbf{Z}(0) = \mathbf{e}_i) = (n + 1)^{-1/\alpha_i} l_i(n), \quad n \to \infty,$$

where $l_i(n)$ is a slowly varying function as $n \to \infty$.

Three types of behavior emerge for the reduced trees, depending on the relative values of α_1 and α_2:

$$1) \quad 1 + \frac{1}{\alpha_1} > \frac{1}{\alpha_2},$$

$$2) \quad 1 + \frac{1}{\alpha_1} < \frac{1}{\alpha_2},$$

$$3) \quad 1 + \frac{1}{\alpha_1} = \frac{1}{\alpha_2}.$$

Three types of behavior emerge for the reduced trees, depending on the relative values of α_1 and α_2:

$$1) \quad 1+\frac{1}{\alpha_1} > \frac{1}{\alpha_2} \Leftrightarrow \mathbf{P}\left(Z_1(n) > 0 | \mathbf{Z}(0) = \mathbf{e}_1\right) = o\left(\mathbf{P}\left(\mathbf{Z}(n) \neq \mathbf{0} | \mathbf{Z}(0) = \mathbf{e}_1\right)\right)$$

$$2) \quad 1+\frac{1}{\alpha_1} < \frac{1}{\alpha_2} \Leftrightarrow \mathbf{P}\left(Z_1(n) > 0 | \mathbf{Z}(0) = \mathbf{e}_1\right) \sim \mathbf{P}\left(\mathbf{Z}(n) \neq \mathbf{0} | \mathbf{Z}(0) = \mathbf{e}_1\right),$$

$$3) \quad 1+\frac{1}{\alpha_1} = \frac{1}{\alpha_2} \Leftrightarrow \mathbf{P}\left(Z_1(n) > 0 | \mathbf{Z}(0) = \mathbf{e}_1\right) \sim K\mathbf{P}\left(\mathbf{Z}(n) \neq \mathbf{0} | \mathbf{Z}(0) = \mathbf{e}_1\right)$$

for some $K \in (0,1)$

Assume that $\alpha_1 = 1$ (the first type has **finite variance for the offspring size**) and

$$1 + \frac{1}{\alpha_1} = 2 < \frac{1}{\alpha_2} \Longrightarrow \alpha_2 \in (0, 1/2)\,.$$

Then
1) If

$$m = \left[\left(1 - e^{-t}\right) n\right], 0 < t < \infty,$$

then, as $n \to \infty$

$$\mathcal{L}\left((Z_1(m,n), Z_2(m,n)) | \mathbf{Z}(n) \neq \mathbf{0}\right) \Longrightarrow \mathcal{L}\left((\zeta(t), 0), 0 < t < \infty\right),$$

where $\zeta(t), \zeta(0) = 1$ is a **single-type continuous time BP** in which particles have exponential life-length which dying produce exactly TWO individuals.

Thus, no particles of the second type within this time-interval.

2) ZOOM 1 If

$$m = n - n^v, \frac{\alpha_2}{1-\alpha_2} < v < 1,$$

then

$$\mathcal{L}\left(\frac{Z_1(m,n)}{n^{1-v}}, Z_2(m,n)|\mathbf{Z}(n) \neq \mathbf{0}\right) \Longrightarrow \mathcal{L}\left((\theta_1(v), 0), \frac{\alpha_2}{1-\alpha_2} < v < 1\right),$$

where the limiting process $\theta_1(v)$ has a.s. constant paths θ_1 with

$$\mathbf{E}e^{-\lambda\theta_1} = \frac{1}{1+\lambda}, \lambda \geq 0.$$

Still no particles of the second type within this time-interval.

3) ZOOM 2 If

$$m = n - tn^{\frac{\alpha_2}{1-\alpha_2}},\ 0 < t < \infty$$

then

$$\mathcal{L}\left(\frac{Z_1(m,n)}{n^{1-\frac{\alpha_2}{1-\alpha_2}}}, Z_2(m,n)\middle|\mathbf{Z}(n) \neq \mathbf{0}\right) \Longrightarrow \mathcal{L}\left(G_1(t), G_2(t); 0 < t < \infty\right),$$

where $(G_1(t), G_2(t))$ is a Markov process with $(G_1(0), G_2(0)) = (\theta_1, 0)$ and

$$\mathbf{E}\left[e^{-\lambda_1 G_1(t)} s_2^{G_2(t)}\right] = \frac{1}{1 + \lambda_1 + t^{1/\alpha_2 - 1}\left(1 - s_2\right)^{1-\alpha_2}}$$

with known transition probabilities.

Coexistence of two types (branching processes with continuous and discrete type spaces).

4) ZOOM 3 If

$$m = n - n^v, 0 < v < \frac{\alpha_2}{1-\alpha_2}$$

then

$$\mathcal{L}\left(\frac{Z_1(m,n)}{n^{1-v}}, \frac{Z_2(m,n)}{n^{\frac{1}{1-\alpha_2}-\frac{v}{\alpha_2}}} | \mathbf{Z}(n) \neq \mathbf{0}\right) \Longrightarrow \mathcal{L}\left((\theta_1, \theta_2), 0 < v < \frac{\alpha_2}{1-\alpha_2}\right)$$

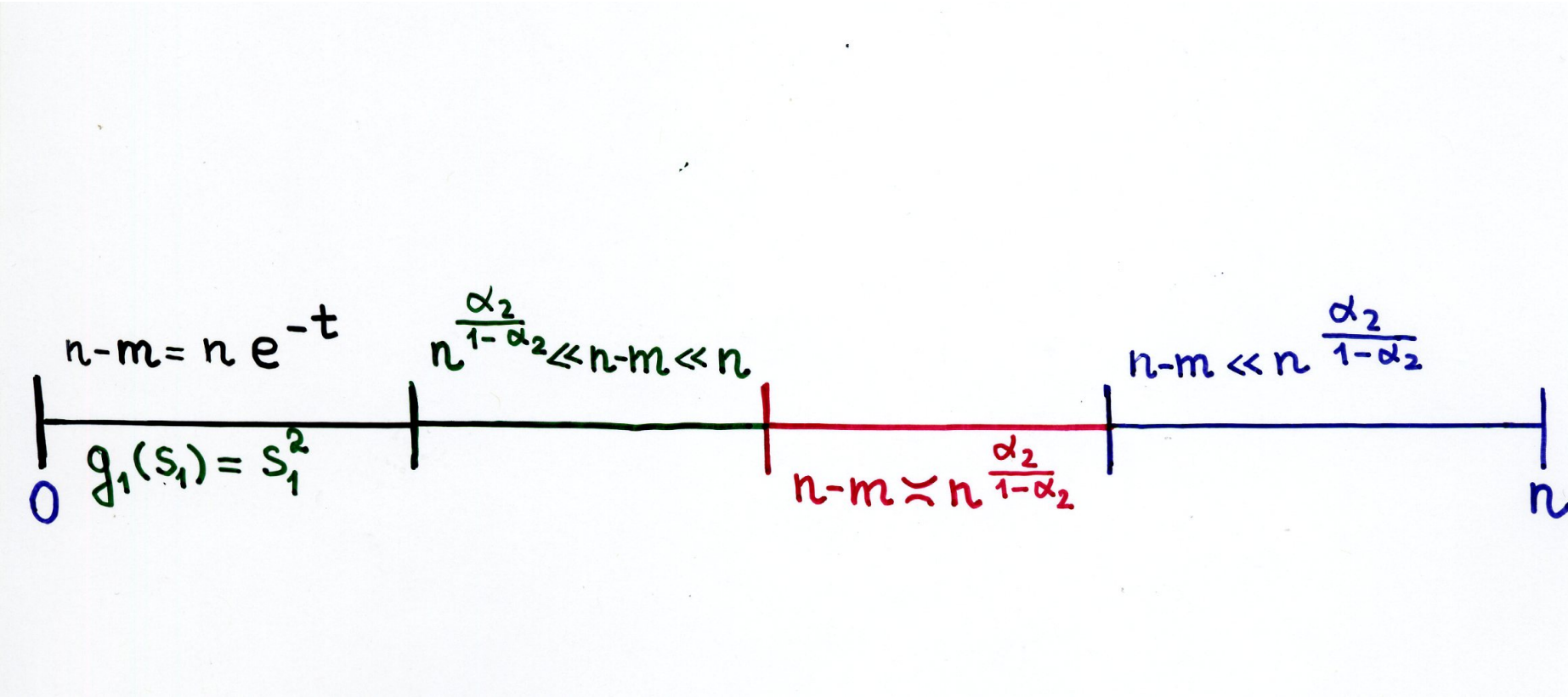
where the limiting process has a.s. constant paths with value (θ_1, θ_2) specified by

$$\mathbf{E}\left[e^{-\lambda_1\theta_1 - \lambda_2\theta_2}\right] = \frac{1}{1+\lambda_1+\lambda_2^{1-\alpha_2}}.$$

Coexistence all the time after the first appearance of the second type. The interval of coexistence is of order

$$n^{\frac{\alpha_2}{1-\alpha_2}}.$$

Coexistence



For the **intermediate case** with $\alpha_1 = 1$ we have

$$1 + \frac{1}{\alpha_1} = 2 = \frac{1}{\alpha_2} \Longrightarrow \alpha_2 = 1/2.$$

Then we have the following asymptotics for the intermediate case
If

$$m = \left[(1 - e^{-t})n\right]$$

then

$$\mathcal{L}\{(Z_1(m,n), Z_2(m,n)), 0 \leq t < \infty\} \Longrightarrow \mathcal{L}\{\mathbf{W}(t), 0 \leq t < \infty\}$$

where the limiting process is a time homogeneous two-type branching process.

The process

$$\mathbf{W}(t) = (W_1(t), W_2(t), 0 \leq t < \infty)$$

starts by a single individual of type 1.

- All particles of the first type have an exponential life-time distribution with some parameter and at the end of their lifes produce either **two children of the first type** with probability

  $$p > 1/2$$

  or **one child of the second type** with probability $1 - p$.
- Particles of the second type have exponential life length and at the end produce at least two children of **the second type only** with PGF

  $$g_2(s_2) = 1 - 3(1 - s_2) + 2(1 - s_2)^{3/2}$$

  satisfying $g_2(0) = g_2'(0) = 0$.

If $n - m = n^v$, where $0 \leq v < 1$, then

$$\mathcal{L}^{(n)} \left\{ \frac{Z_1(m,n)}{n^{1-v}}, \frac{Z_2(m,n)}{n^{(1-v)/\alpha_2}}, 0 \leq v < 1 \right\} \Longrightarrow \mathcal{L}\{\mathbf{\Upsilon}(v), 0 \leq v < 1\},$$

where the limiting process has a.s. constant paths with value (ζ_1, ζ_2) specified by

$$\mathbf{E}\left[e^{-\lambda_1 \zeta_1} e^{-\lambda_2 \zeta_2}\right] = 1 - \psi\left(\lambda_1, \lambda_2\right)$$

and

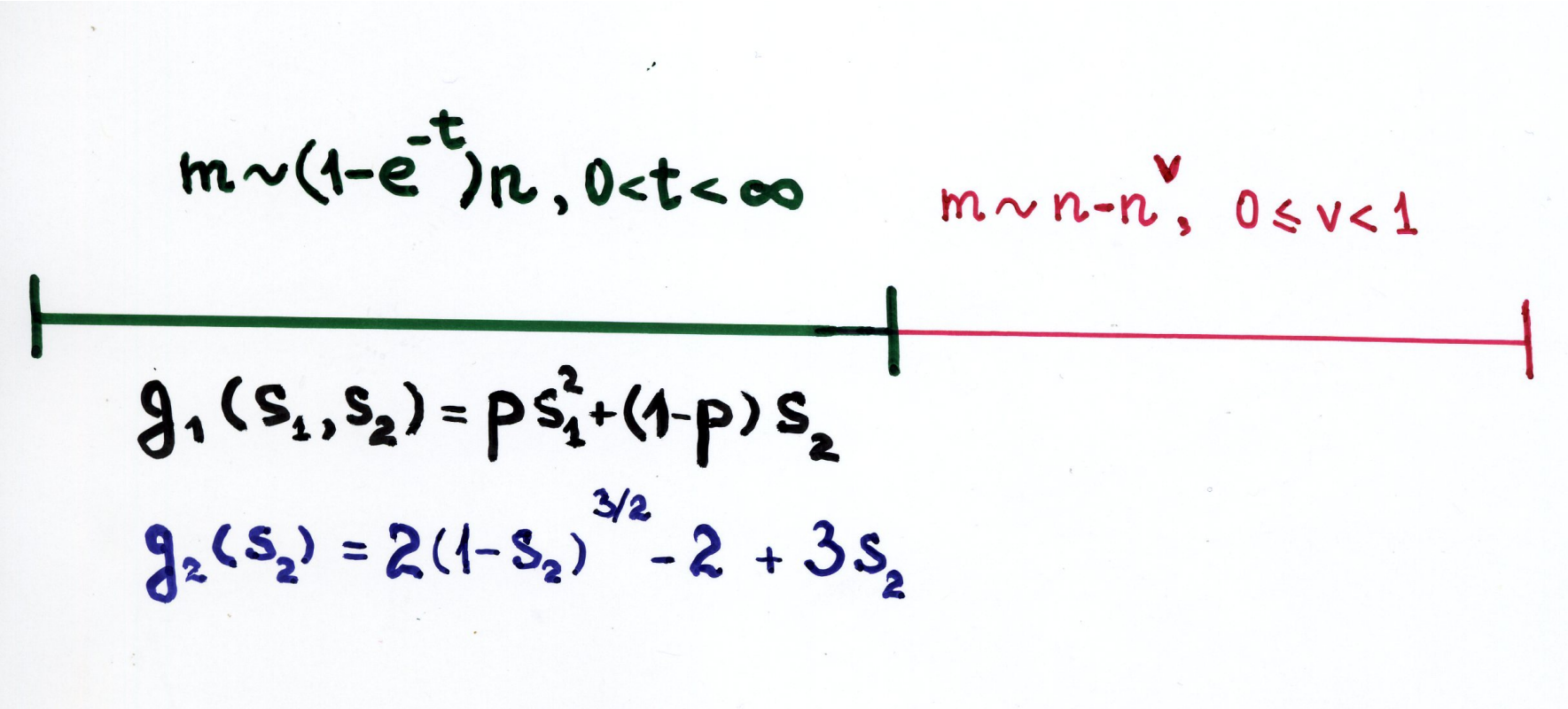
$$\lambda_1 \frac{\partial \psi}{\partial \lambda_1} + 2\lambda_2 \frac{\partial \psi}{\partial \lambda_2} = \psi - b\psi^2 + \frac{A_{12}\lambda_2}{\left(1 + C\sqrt{\lambda_2}\right)^2}$$

with initial conditions

$$\psi\left(0,0\right) = 0, \quad \frac{\partial \psi\left(\lambda_1, \lambda_2\right)}{\partial \lambda_1} |_{\lambda_1 = \lambda_2 = 0} = 1, \quad \frac{\partial \psi\left(\lambda_1, \lambda_2\right)}{\partial \lambda_2} |_{\lambda_1 = \lambda_2 = 0} = A_{12}.$$

In this case two types of particles may coexist during an interval of order n.

Reduced process



THANKS!